Fall 2012 Math 152

Week in Review 6

courtesy: *Oksana Shatalov*

(covering Section 9.3, 9.4 &10.1)

9.3: Arc Length

Key Points

- Arc length of curve C: $\displaystyle\int \mathrm{d}s$
- $\displaystyle\int \mathrm{d}s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ when C is given by $x = x(t), y = y(t), \alpha \leq t \leq \beta$;
- $\displaystyle\int \mathrm{d}s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ when C is given by $y = y(x)$, $a \leq x \leq b$;
- $\displaystyle\int \mathrm{d}s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$ when C is given by $x = x(y)$, $c \leq y \leq d$.

1. Find the length of the curve $x = \cos^3 t, y = \sin^3 t$, $0 \le t \le \pi/2$.

$$L = \int ds = \int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$x'(t) = (\cos^3 t)' = 3\cos^2 t \cdot (-\sin t) = -3\cos^2 t \sin t$$

$$y'(t) = (\sin^3 t)' = 3\sin^2 t \cos t$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{=1}} = 3|\cos t \sin t|$$

positive for given value of t

$$= 3\cos t \sin t$$

Plug in:

$$L = \int_0^{\pi/2} \underbrace{3\cos t \sin t}_{\frac{\sin 2t}{2}}\, dt = \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt$$

$$= \frac{3}{2}\left(-\frac{\cos 2t}{2}\right)\Bigg|_0^{\pi/2}$$

$$= -\frac{3}{4}(\cos\pi - \cos 0) = -\frac{3}{4}(-1-1) = \boxed{\frac{3}{2}}$$

Note: You can do it using u-sub.

2. Find the length of the curve $x = \frac{1}{4}\ln y - \frac{1}{2}y^2$ from $y = 1$ to $y = e$.

$x = x(y)$

$$L = \int ds = \int_1^e \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{4y} - y = \frac{1-4y^2}{4y}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{(1-4y^2)^2}{(4y)^2} = \frac{16y^2 + 1 - 8y^2 + 16y^4}{(4y)^2}$$

$$= \frac{1 + 8y^2 + 16y^4}{(4y)^2} = \frac{(1+4y^2)^2}{(4y)^2}$$

$$L = \int dS = \int_1^e \sqrt{\frac{(1+4y^2)^2}{(4y)^2}}\, dy = \int_1^e \frac{1+4y^2}{4y}\, dy$$

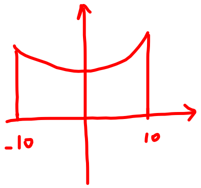
$$= \int_1^e \frac{1}{4y} + y\, dy = \frac{1}{4}\ln y + \frac{y^2}{2}\Big|_1^e$$

$$= \frac{1}{4}\underbrace{\ln e}_{=1} + \frac{e^2}{2} - \frac{1}{4}\underbrace{\ln 1}_{=0} - \frac{1}{2} = \boxed{\frac{e^2}{2} - \frac{1}{4}}$$

3. A wire hanging between two poles (at $x = -10$ and $x = 10$) takes the shape of a catenary with equation

$$y = 2(e^{x/4} + e^{-x/4}).$$

Find the length of the wire.



Find arc length of

$y = 2\left(e^{x/4} + e^{-x/4}\right)$, $-10 \le x \le 10$

$$L = \int ds = \int_{-10}^{10} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = 2\left(\frac{1}{4}e^{x/4} - \frac{1}{4}e^{-x/4}\right) = \frac{1}{2}\left(e^{\frac{x}{4}} - e^{-\frac{x}{4}}\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{x/4} - e^{-x/4}\right)^2 = \frac{1}{4}\left(\left(e^{\frac{x}{4}}\right)^2 - 2\underbrace{e^{\frac{x}{4}} \cdot e^{-\frac{x}{4}}}_{=1} + \left(e^{-\frac{x}{4}}\right)^2\right)$$

$$= \frac{1}{4}\left(e^{\frac{x}{2}} - 2 + e^{-\frac{x}{2}}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(e^{\frac{x}{2}} - 2 + e^{-\frac{x}{2}}\right) = \frac{4 + e^{\frac{x}{2}} - 2 + e^{-\frac{x}{2}}}{4}$$

$$= \frac{e^{\frac{x}{2}} + 2 + e^{-\frac{x}{2}}}{4} = \frac{\left(e^{\frac{x}{4}} + e^{-x/4}\right)^2}{4}$$

$$L = \int_{-10}^{10} \sqrt{\frac{\left(e^{x/4} + e^{-x/4}\right)^2}{4}}\, dx = \frac{1}{2}\int_{-10}^{10} e^{\frac{x}{4}} + e^{-\frac{x}{4}}\, dx$$

$$= \frac{1}{2}\left(4e^{\frac{x}{4}} - 4e^{-\frac{x}{4}}\right)\Big|_{-10}^{10} = 2\left(e^{\frac{x}{4}} - e^{-\frac{x}{4}}\right)\Big|_{-10}^{10} =$$

$$= 2\left(e^{5/2} - e^{-5/2} - \left(e^{-5/2} - e^{5/2}\right)\right)$$

$$= 2\left(e^{5/2} - e^{-5/2} - e^{-5/2} + e^{5/2}\right)$$

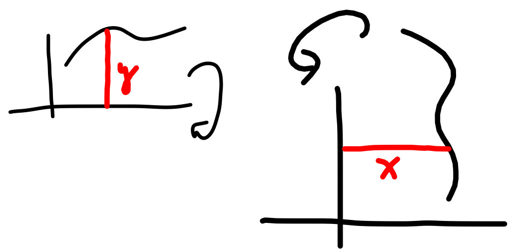
$$= 2\left(2e^{5/2} - 2e^{-5/2}\right) = 4\left(e^{5/2} - e^{-5/2}\right)$$

9.4: Area of a Surface of Revolution

Key Points

distance to axis of rotating

- $SA = 2\pi \int (radius) ds$
- about the x-axis: $radius = y$
- about the y-axis: $radius = x$



4. The curve $y = x^2$, $0 \le x \le 1$, is rotated about the y-axis. Find the area of the resulting surface.

$$SA = 2\pi \int x\, ds = 2\pi \int_0^1 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(\frac{dy}{dx}\right)^2 = (2x)^2 = 4x^2$$

$$= 2\pi \int_0^1 x \sqrt{1 + 4x^2}\, dx$$

u-sub. $u = 1 + 4x^2$, $du = 8x\, dx$ | $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 5$

$$SA = 2\pi \int_1^5 \sqrt{u}\, \frac{du}{8} = \frac{\pi}{4} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \frac{2}{3} u^{3/2} \Big|_1^5 = \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

5. The curve $x = 1 - \cos(2t)$, $y = 2t + \sin(2t)$, $0 \le t \le \pi/4$ is rotated about the x-axis. Find the area of the resulting surface.

$$SA = 2\pi \int y\, ds = 2\pi \int_0^{\pi/4} \left(2t + \sin(2t)\right) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$x'^2 + y'^2 = (2\sin(2t))^2 + (2 + 2\cos(2t))^2$$

$$= 4\sin^2(2t) + 4 + 2\cdot 2\cdot 2\cos(2t) + 4\cos^2(2t)$$

$$= 4\underbrace{\left(\sin^2(2t) + \cos^2(2t)\right)}_{=1} + 8\cos 2t + 4$$

Hidden square

$$= 4 + 8\cos 2t + 4 = 8 + 8\cos 2t = 8(1 + \cos 2t)$$

Use $\cos^2 t = \frac{1+\cos 2t}{2}$

$$= 8 \cdot 2\cos^2 t = (4\cos t)^2$$

$$SA = 2\pi \int_0^{\pi/4} (2t + \sin 2t)\sqrt{(4\cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi/4} (2t + \sin 2t)\cdot 4\cos t\, dt$$

$$= 8\pi \left[\int_0^{\pi/4} 2t\cos t\, dt + \int_0^{\pi/4} \sin 2t \cos t\, dt \right]$$

INTEGR. BY PARTS

$u = 2t$, $dv = \cos t\, dt$

$du = 2\, dt$, $v = \sin t$

$$2t\sin t\Big|_0^{\pi/4} - 2\int_0^{\pi/4} \sin t\, dt$$

$$= 2\cdot\frac{\pi}{4}\sin\frac{\pi}{4} - 0 + 2\cos t\Big|_0^{\pi/4}$$

$$= \frac{\pi}{2}\frac{\sqrt{2}}{2} + 2\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$= \frac{\pi\sqrt{2}}{4} + \sqrt{2} - 2$$

$$= \int_0^{\pi/4} 2\cos t \sin t \cos t\, dt$$

$$= 2\int_0^{\pi/4} \cos^2 t \sin t\, dt$$

$u = \cos t \Rightarrow du = -\sin t\, dt$

$t = 0 \Rightarrow u = \cos 0 = 1$

$t = \frac{\pi}{4} \Rightarrow u = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$= 2\int_1^{\sqrt{2}/2} u^2(-du) = 2\int_{\sqrt{2}/2}^{1} u^2\, du$$

$$= \frac{2u^3}{3}\Big|_{\frac{\sqrt{2}}{2}}^{1} = \frac{2}{3}\left(1 - \frac{2\sqrt{2}}{2^3}\right) = \frac{2}{3}\left(1 - \frac{\sqrt{2}}{4}\right)$$

$$= \frac{2}{3} - \frac{\sqrt{2}}{6}$$

$$SA = 8\pi\left[\frac{\pi\sqrt{2}}{4} + \sqrt{2} - 2 + \frac{2}{3} - \frac{\sqrt{2}}{6}\right]$$

$$= 8\pi\left[\frac{\pi\sqrt{2}}{4} + \frac{5\sqrt{2}}{6} - \frac{4}{3}\right]$$

6. Set up (but don't evaluate) the integral that gives the surface area obtaine by rotating the curve

$$x = \sin(\pi y^2/8), \quad 1 \le y \le 2,$$

(a) about the x-axis

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$SA = 2\pi \int y\, ds$$

$$ds = \sqrt{1 + \left(\frac{2\pi y}{8} \cos\frac{\pi y^2}{8}\right)^2}\, dy$$

$$SA = 2\pi \int_1^2 y \sqrt{}\, dy$$

$$ds = \sqrt{1 + \frac{\pi^2 y^2}{16} \cos^2\frac{\pi y^2}{8}}\, dy$$

(b) about the y-axis

$$SA = 2\pi \int x\, ds = 2\pi \int_1^2 \sin\frac{\pi y^2}{8} \sqrt{}\, dy$$

7. The curve $x = \sin(at), y = \cos(at), 0 \le t \le \frac{\pi}{2a}$ is rotated about the x-axis (here a is an arbitrary positive constant). Find the area of the resulting surface.

$$SA = 2\pi \int y\, ds = 2\pi \int_0^{\pi/2a} \cos(at) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$SA = 2\pi \int_0^{\pi/2a} \cos(at) \sqrt{[a\cos(at)]^2 + [-a\sin(at)]^2}\, dt$$

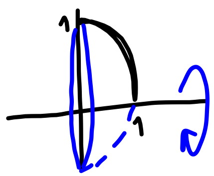
$$SA = 2\pi \int_0^{\pi/2a} \cos at \sqrt{a^2 \big(\underbrace{\cos^2(at) + \sin^2(at)}_{1}\big)}\, dt$$

$$SA = 2\pi \int_0^{\pi/2a} a\cos(at)\, dt = 2\pi \sin(at) \Big|_0^{\frac{\pi}{2a}}$$

$$= 2\pi \left(\sin\left(\cancel{a} \cdot \frac{\pi}{2\cancel{a}}\right) - \sin 0\right) = \boxed{2\pi}$$

Note The given curve is quarter of unit circle

$$SA(\text{sphere}) = 4\pi R^2$$



$$SA\left(\tfrac{1}{2}\text{sphere}, r=1\right) = \frac{4\pi \cdot 1^2}{2} = \boxed{2\pi}$$

10.1: Sequences

Key Points

- If $\lim_{n\to\infty} a_n$ exists and finite then we say that the sequence $\{a_n\}$ **converges**. Otherwise, we say the sequence **diverges**. (Recall all techniques for finding limits at infinity.)
- The *Squeeze* Theorem for Sequences: If $a_n \le b_n \le c_n$ for all n and the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to \infty$, then $\lim_{n\to\infty} b_n = L$.
- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
- $\{a_n\}$ *increasing:* show that $a_{n+1} - a_n > 0$, or $f'(x) > 0$ (where $f(n) = a_n$); or $\dfrac{a_n+1}{a_n} > 1$ (provided $a_n > 0$ for all n.) Note: reverse signs for $\{a_n\}$ *decreasing*.

8. Define the n-th term of the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\right\}$ and find its limit.

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$$\boxed{a_n = \frac{n}{n+1}}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \boxed{1}$$

9. Determine if the given sequence converges or diverges. If it converges, find the limit.

Rational function

(a) $a_n = \dfrac{3n^5 - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 - 9n^5}$

(b) $b_n = \dfrac{3n^5 - 12n^3 + 2012}{2012 - 12n^4 - 4n^4 - 9n^5 + 11n^6}$

(c) $c_n = \dfrac{12n^7 + 2012}{2012 - 12n^4 - 4n^5 - 9n^6}$

	deg (num)		deg(den.)
(a)	5	=	5
(b)	5	<	6
(c)	7	>	6

(a) $\lim_{n\to\infty} a_n = \dfrac{3}{-9} = \boxed{-\dfrac{1}{3}}$; (b) $\lim_{n\to\infty} b_n = \boxed{0}$; (c) $\lim_{n\to\infty} c_n = -\infty$, c_n is $\boxed{\text{divergent}}$

10. Determine if the sequence with the given general term ($n \geq 1$) converges or diverges. If it converges, find the limit.

(a) $a_n = \ln(n^2+3) - \ln(7n^2-5) = \ln \frac{n^2+3}{7n^2-5}$ $\qquad \frac{n^2+3}{7n^2-5} \xrightarrow[n\to\infty]{} \frac{1}{7}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{n^2+3}{7n^2-5} = \ln \frac{1}{7}$$

(b) $z_n = \frac{1}{n^4}\sin(\frac{1}{n^5})$ $\qquad$ Note that $-1 \leq \sin x \leq 1$

Way 1

$$\frac{1}{n^4}(-1) \leq \frac{1}{n^4} \sin \frac{1}{n^5} \leq \frac{1}{n^4}(1)$$

$$-\frac{1}{n^4} \leq \frac{1}{n^4} \sin \frac{1}{n^5} \leq \frac{1}{n^4}$$

$n\to\infty$: $-\frac{1}{n^4} \to 0$, $\frac{1}{n^4} \to 0$

By Squeeze Theorem $\frac{1}{n^4} \sin \frac{1}{n^5} \to 0$

$$\boxed{\lim_{n\to\infty} z_n = 0}$$

Way 2 $\quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1}{n^4} \sin \frac{1}{n^5} = 0 \cdot \sin(0) = 0$

(b') $z_n = \frac{1}{n^4} \sin n^5$

"Way 2" from (b) is not applicable here.

Use Squeeze Theorem:

$$-\frac{1}{n^4} \leq \frac{1}{n^4} \sin n^5 \leq \frac{1}{n^4}$$

$-\frac{1}{n^4} \to 0$, $\frac{1}{n^4} \to 0$, $\frac{1}{n^4}\sin n^5 \xrightarrow[n\to\infty]{} 0$

$\lim z_n = 0$

(c) $y_n = \frac{(-1)^n}{n^3}$

$$\lim_{n\to\infty} |y_n| = \lim_{n\to\infty} \frac{1}{n^3} = 0 \Rightarrow \lim_{n\to\infty} y_n = \boxed{0}$$

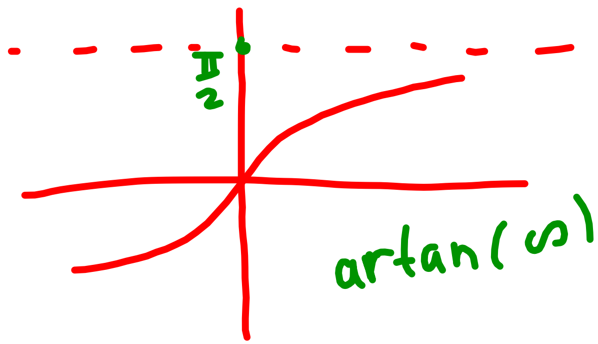
(d) $x_n = \dfrac{(-1)^n n}{3n+33}$

If n is even $\Rightarrow$ $x_n = \dfrac{n}{3n+33} \xrightarrow[n\to\infty]{} \dfrac{1}{3}$

If n is odd $\Rightarrow$ $x_n = -\dfrac{n}{3n+33} \xrightarrow[n\to\infty]{} -\dfrac{1}{3}$

$\Big\}$ $\lim\limits_{n\to\infty} x_n$ DNE

(e) $a_n = \dfrac{(\arctan n)^7}{n^5}$



$\text{artan}(\infty) = \dfrac{\pi}{2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(\frac{\pi}{2}\right)^7}{n^5} = 0$$

11. Assuming that the sequence defined recursively by $a_n = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$ is convergent, find its limit.

Note that $\{a_n\}$ is positive $\Rightarrow L > 0$

Denote $L = \lim_{n\to\infty} a_n$

$$L = \lim_{n\to\infty} a_{n+1} = \frac{1}{2} \lim_{n\to\infty} \left(a_n + \frac{9}{a_n}\right) = \frac{1}{2}\left(L + \frac{9}{L}\right)$$

$$L = \frac{1}{2}\left(L + \frac{9}{L}\right)$$

$$2L = L + \frac{9}{L} \Rightarrow 2L = \frac{L^2 + 9}{L}$$

$$2L^2 = L^2 + 9$$

$$L^2 = 9$$

$$L = \pm 3$$

$$\boxed{L = 3}$$

12. Determine whether the given seqence is increasing or decreasing.

(a) $\{\arctan(n)\}_{n=1}^{\infty}$

Define $f(x)$ such that $f(n) = \arctan(n)$, i.e.

$f(x) = \arctan(x)$, $x \geq 1$

$f'(x) = \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} > 0$

$f(x) \uparrow$ for $x \geq 1$ $\Rightarrow$ $\{\arctan(n)\}_{n=1}^{\infty}$ is $\boxed{\text{increasing}}$

(b) $\{n - 2^n\}_{n=1}^{\infty}$ $\quad a_n = n - 2^n$, $a_{n+1} = n+1-2^{n+1}$

$a_{n+1} - a_n = n+1-2^{n+1} - (n-2^n) = \cancel{n}+1-2^{n+1}-\cancel{n}+2^n$

$= 1-2^{n+1}+2^n = 1-2\cdot 2^n + 2^n = 1-2^n < 0$ for all n

$\{n-2^n\}$ is decreasing

$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n$

(c) $\left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty}$ $\quad$ Note $\frac{10^n}{n!} > 0 \Rightarrow \frac{a_{n+1}}{a_n} \overset{?}{>} 1$

$a_n = \frac{10^n}{n!}$ $\qquad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{\cancel{10^n}\cdot 10\cdot \cancel{n!}}{\cancel{n!}\,(n+1)\cdot\cancel{10^n}} = \frac{10}{n+1}$

$(n+1)! = \underbrace{1\cdot 2\cdot\ldots\cdot n}_{n!}\cdot(n+1)$

$\boxed{(n+1)! = n!\,(n+1)}$

$\frac{10}{n+1} < 1 \Rightarrow 10 < n+1 \Rightarrow \underbrace{n > 9}$ decreasing

$\frac{10}{n+1} \geq 1 \Rightarrow n \leq 9$ increasing

$\boxed{\text{not monotonic}}$