Fall 2012 Math 152

Week in Review 9

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(covering Sections 10.5& 10.6)

10.5: Power Series

Key Points

- For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:
  1. There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the **radius of convergence**.
  2. The series converges only for $x = a$ (then $R = 0$).
  3. The series converges for all x (then $R = \infty$).
- We find the radius of convergence using the **Ratio Test.** $\lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$
- An **interval of convergence** is the interval of all x's for which the power series converges.
- You must check the endpoints $x = a \pm R$ individually to determine whether or not they are in the interval of convergence.

$$|x-a| < R \Rightarrow -R < x-a < R$$

$$a-R < x < a+R$$

1. For the following series find the radius and interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^4 x^n}{7^n}$

$$|a_n| = \frac{n^4 |x|^n}{7^n}$$

$$|a_{n+1}| = \frac{(n+1)^4 |x|^{n+1}}{7^{n+1}}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^4 |x|^{\cancel{n+1}}}{7^{\cancel{n+1}}} \cdot \frac{\cancel{7^n}}{n^4 \cancel{|x|}}$$

$$= \frac{|x|}{7} \underbrace{\lim_{n\to\infty} \frac{(n+1)^4}{n^4}}_{1} = \frac{|x|}{7} < 1 \Rightarrow |x| < 7$$

$$\Downarrow \quad \boxed{R = 7}$$

L=1 To find interval of convergence test end points:

$|x| = 7$ OR

$x = 7 \Rightarrow \displaystyle\sum_{n\geq 0}^{\infty} \frac{n^4 7^n}{7^n} = \sum n^4$ diverges by DT: $\lim_{n\to\infty} n^4 = \infty$

$x = -7 \Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{n^4 (-7)^n}{7^n} = \sum_{n=0}^{\infty} (-1)^n n^4$ diverges by Div. Test

$(-1)^n \cdot 7^n$

$\lim_{n\to\infty} (-1)^n n^4$ DNE

$$\boxed{(-7, 7)}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{8^n (x+4)^{3n}}{n^3+1}$

$$|a_n| = \frac{8^n |x+4|^{3n}}{n^3+1}$$

$$|a_{n+1}| = \frac{8^{n+1} |x+4|^{3(n+1)}}{(n+1)^3+1}$$

$(3n+3)$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{8^{\cancel{n+1}} |x+4|^{\cancel{3n+3}}}{(n+1)^3+1} \cdot \frac{n^3+1}{\cancel{8^n} \cancel{|x+4|^{3n}}}$$

$$= 8|x+4|^3 \underbrace{\lim_{n\to\infty} \frac{n^3+1}{(n+1)^3+1}}_{=1} = 8|x+4|^3 < 1$$

$$|x+4|^3 < \frac{1}{8}$$

$$|x+4| < \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$$

$$\boxed{R = \frac{1}{2}}$$

$L=1$ (end points): $|x+4| = \frac{1}{2}$

$x+4 = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{8^n \left(\frac{1}{2}\right)^{3n}}{n^3+1}$$

$x+4 = -\frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{8^n \left(-\frac{1}{2}\right)^{3n}}{n^3+1}$$

$$\sum_{n=0}^{\infty} \frac{8^n \cdot \frac{(-1)^{3n}}{8^n}}{n^3+1}$$

$$\left(\frac{1}{2}\right)^{3n} = \left(\left(\frac{1}{2}\right)^3\right)^n = \frac{1}{8^n}$$

$$\left(-\frac{1}{2}\right)^{3n} = (-1)^{3n} \cdot \left(\frac{1}{2}\right)^{3n} = \frac{(-1)^{3n}}{8^n}$$

$$\sum_{n=0}^{\infty} \frac{8^n \cdot \frac{1}{8^n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{1}{n^3+1}$$

converges by Comparison Test

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

and $\sum \frac{1}{n^3}$ convergent ($p = 3 > 1$)

$$\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{3n}}{n^3+1}}_{\text{by}}$$

the series converges absolutely

$\Downarrow$

convergent

Finally, INT. of CONV:

$$|x+4| \leq \frac{1}{2}$$

$$-\frac{1}{2} \leq x+4 \leq \frac{1}{2}$$

$$-\frac{1}{2} - 4 \leq x \leq \frac{1}{2} - 4$$

$$\boxed{-\frac{9}{2} \leq x \leq -\frac{7}{2}}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n(5x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n 9^n 5^n (x-\frac{3}{5})^n}{n}$

Note $(5x-3)^n = 5^n(x-\frac{3}{5})^n$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 45^n}{n}(x-\tfrac{3}{5})^n$$

$|a_n| = \frac{45^n}{n}|x-\frac{3}{5}|^n$, $|a_{n+1}| = \frac{45^{n+1}|x-\frac{3}{5}|^{n+1}}{n+1}$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{45^{\cancel{n+1}}|x-\frac{3}{5}|^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{45^n}\,\cancel{|x-\frac{3}{5}|^n}}$$

$$= 45|x-\tfrac{3}{5}| \lim_{n\to\infty} \frac{n}{n+1} = 45|x-\tfrac{3}{5}| < 1$$

$|x-\frac{3}{5}| < \frac{1}{45} \Rightarrow \boxed{R = \frac{1}{45}}$

$L=1 \Rightarrow |x-\frac{3}{5}| = \frac{1}{45}$

$x - \frac{3}{5} = \frac{1}{45}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 45^n}{n} \cdot \left(\frac{1}{45}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ convergent by AST!

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\{\frac{1}{n}\} \downarrow$

$x - \frac{3}{5} = -\frac{1}{45}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 45^n}{n} \cdot \left(-\frac{1}{45}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}\,\cancel{45^n}}{n} \frac{1}{\cancel{(-1)^n}\cdot \cancel{45^n}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}}$$

$p=1$

$$-\frac{1}{45} < x - \frac{3}{5} \le \frac{1}{45}$$

$$-\frac{1}{45} + \overset{9}{\cancel{\tfrac{3}{5}}} < x \le \frac{1}{45} + \overset{9}{\cancel{\tfrac{3}{5}}}$$

$\boxed{\frac{26}{45} < x \le \frac{28}{45}}$ or $\left(\frac{26}{45}, \frac{28}{45}\right]$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!(x-1)^{n+1}}{4^{n+1}}$

$$|a_n| = \frac{(n+1)!\ |x-1|^{n+1}}{4^{n+1}}$$

$$|a_{n+1}| = \frac{(n+2)!\ |x-1|^{n+2}}{4^{n+2}}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(n+2)!\ |x-1|^{n+2}}{4^{n+2}} \cdot \frac{4^{n+1}}{(n+1)!\ |x-1|^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}\,(n+2)\cdot \cancel{|x-1|^{n+1}}\cdot |x-1|}{\cancel{4^{n+1}}\cdot 4} \cdot \frac{\cancel{4^{n+1}}}{\cancel{(n+1)!}\ \cancel{|x-1|^{n+1}}}$$

$$= \frac{|x-1|}{4} \lim_{n\to\infty} (n+2) = \begin{cases} \infty > 1, & x \neq 1 \quad \text{divergent} \\ 0 < 1, & x = 1 \quad \text{convergent} \end{cases}$$

$R = 0$

int. of conv. $\{1\}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-6)^n x^n}{(3n+1)!}$

$$|a_n| = \frac{6^n |x|^n}{(3n+1)!}$$

$$|a_{n+1}| = \frac{6^{n+1} |x|^{n+1}}{(3(n+1)+1)!} = \frac{6^{n+1} |x|^{n+1}}{(3n+4)!}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{6^{\cancel{n+1}} \cdot |x|^{\cancel{n+1}}}{(3n+4)!} \cdot \frac{(3n+1)!}{\cancel{6^n} \cdot \cancel{|x|^n}}$$

$$= 6|x| \lim_{n\to\infty} \frac{\cancel{(3n+1)!}}{\cancel{(3n+1)!} \cdot (3n+2)(3n+3)(3n+4)} = 0 < 1$$

for all x

$$\boxed{R = \infty \;,\; (-\infty, \infty)}$$

2. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges when $x = 5$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following series:

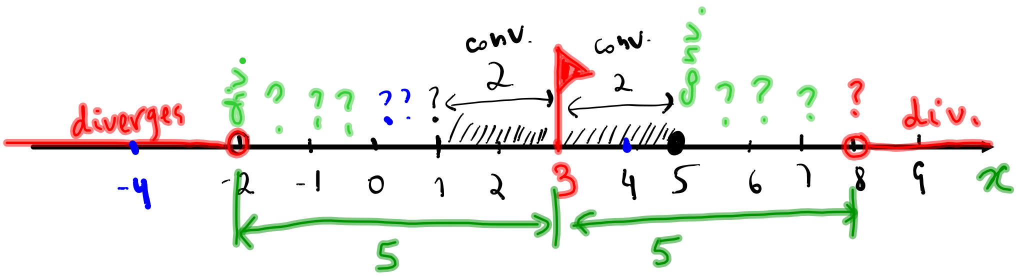
(a) $\sum_{n=0}^{\infty} c_n(-7)^n$ $\quad x-3=-7 \Rightarrow x=-7+3=-4$ $\quad$ divergent

(b) $\sum_{n=0}^{\infty} c_n 5^n$ $\quad x-3=5 \Rightarrow x=5+3=8$ $\quad$ don't know

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$ $\quad x-3=-3 \Rightarrow x=-3+3=0$ $\quad$ don't know

(d) $\sum_{n=0}^{\infty} c_n 3^n$ $\quad x-3=3 \Rightarrow x=3+3=6$ $\quad$ don't know

(e) $\sum_{n=0}^{\infty} c_n(-1)^n$ $\quad x-3=-1 \Rightarrow x=-1+3=2$ $\quad$ convergent



10.6: Representation of Functions as Power Series

diverges as $x = \pm 1$ at end points

Key Points

- Geometric Series Formula:

$$\boxed{\frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.}$$

Test end points

- **Term-by-term Differentiation and Integration of power series:**

  If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

  – $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

  – $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

  The radii of convergence of the power series for $f'(x)$ and $\int f(x)\,dx$ are both R.

3. Find a power series representation for the following functions and determine the interval of convergence.

(a) $f(x) = \dfrac{4}{1+x}$ $= 4 \cdot \dfrac{1}{1-(-x)} = 4 \cdot \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 4 \cdot x^n$

$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow (-1, 1)$

we don't need to test end points here

<u>Note</u> $\boxed{\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n}$

(b) $f(x) = \dfrac{4}{2+4x} = \dfrac{2}{1+2x} = 2 \cdot \dfrac{1}{1+2x} = 2 \cdot \dfrac{1}{1-(-2x)}$

$$= 2 \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} \underbrace{2} \cdot (-1)^n \underbrace{2^n} x^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} x^n}$$

$|-2x| < 1$

$|2x| < 1$

$|x| < \frac{1}{2}$

$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

(c) $f(x) = \dfrac{-9}{9 - x^4} = \dfrac{-9}{9\left(1 - \frac{x^4}{9}\right)} = - \dfrac{1}{1 - \frac{x^4}{9}}$

$$f(x) = - \sum_{n=0}^{\infty} \left(\frac{x^4}{9}\right)^n = \boxed{\sum_{n=0}^{\infty} - \frac{x^{4n}}{9^n}}$$

where $\left|\frac{x^4}{9}\right| < 1 \Rightarrow \frac{x^4}{9} < 1 \Rightarrow \sqrt[4]{x^4} < \sqrt[4]{9}$

$|x| < \sqrt[4]{9} = \sqrt[4]{3^2}$

$|x| < \sqrt{3}$

$\boxed{-\sqrt{3} < x < \sqrt{3}}$

(d) $f(x) = \dfrac{x^{2012}}{2012 - x} = x^{2012} \dfrac{1}{2012\left(1 - \frac{x}{2012}\right)}$

$$f(x) = \frac{x^{2012}}{2012} \cdot \boxed{\frac{1}{1 - \frac{x}{2012}}} = \frac{x^{2012}}{2012} \sum_{n=0}^{\infty} \left(\frac{x}{2012}\right)^n$$

sum of geom. series with common ratio $\frac{x}{2012}$

Finally

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2012} \cdot x^n}{2012 \cdot 2012^n} = \sum_{n=0}^{\infty} \frac{x^{n+2012}}{2012^{n+1}}$$

where $\left|\frac{x}{2012}\right| < 1$

$\Downarrow$ $|x| < 2012$

$-2012 < x < 2012$ or $\boxed{(-2012, 2012)}$

(e) $f(x) = \ln(3x+5)$

$$f'(x) = \frac{3}{3x+5} = \frac{3}{5\left(\frac{3}{5}x+1\right)} = \frac{3}{5}\cdot\frac{1}{1-\left(-\frac{3}{5}x\right)} =$$

$$= \frac{3}{5}\sum_{n=0}^{\infty}\left(-\frac{3}{5}x\right)^n = \sum_{n=0}^{\infty}\frac{3}{5}\cdot(-1)^n\cdot\left(\frac{3}{5}\right)^n x^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1} x^n$$

where $\left|-\frac{3}{5}x\right| < 1$

$$\left|\frac{3x}{5}\right| < 1 \Rightarrow |3x| < 5 \Rightarrow |x| < \frac{5}{3} \Rightarrow \underbrace{-\frac{5}{3} < x < \frac{5}{3}}$$

Note that this is the interval of convergence for $f'(x)$ and not for $f(x)$.

To find int. of conv. for $f(x)$ we need to test power series of $f(x)$ at end points $x = \pm 5/3$

$$f(x) = \int f'(x)\,dx = \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\int x^n\,dx$$

$$\ln(3x+5) = f(x) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1} + C$$

To determine C plugin center of power series:

$x=0$ $\quad \ln 5 = f(0) = 0 + C \Rightarrow C = \ln 5$

$$\boxed{f(x) = \ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}}$$

Test end points $x = -\frac{5}{3}$

$$\ln 5 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{n+1}\cdot\frac{\left(-\frac{5}{3}\right)^{n+1}}{n+1}$$

$$= \ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^n\cdot\cancel{\left(\frac{3}{5}\right)^{n+1}}\cdot(-1)^{n+1}\cdot\cancel{\left(\frac{5}{3}\right)^{n+1}}}{n+1}$$

$$= \ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{n+1} = \ln 5 - \underbrace{\sum_{n=0}^{\infty}\frac{1}{n+1}}_{\text{divergent by Integral test}}$$

$$\int_0^{\infty}\frac{dx}{x+1} = \int_1^{\infty}\frac{du}{u} \text{ divergent}$$

$x = \frac{5}{3}$

$$\ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^n\cancel{\left(\frac{3}{5}\right)^{n+1}}\cdot\cancel{\left(\frac{5}{3}\right)^{n+1}}}{n+1}$$

$$= \ln 5 + \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}}_{\text{conv. by AST}}$$

Finally, INTERVAL of convergence

$$\boxed{-\frac{5}{3} < x \leq \frac{5}{3}}$$

(f) $f(x) = x^5 \ln(3x+5)$

By Part (e) we have

$$f(x) = x^5 \cdot \left(\ln 5 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{5}\right)^{n+1} \frac{x^{n+1}}{n+1} \right.$$

where $\boxed{-\frac{5}{3} < x \leq \frac{5}{3}}$

$$f(x) = (\ln 5)\, x^5 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{5}\right)^{n+1} \frac{x^{n+6}}{n+1}$$

(g) $f(x) = \dfrac{x^4}{(1-4x)^2} = x^4 \underbrace{\dfrac{1}{(1-4x)^2}}_{g(x)}$

Find decomp. for $g(x)$ first:

$$g(x) = \frac{1}{(1-4x)^2}$$

$$\int g(x)\,dx = \int \frac{dx}{(1-4x)^2} = \frac{1}{4(1-4x)} + C = \frac{1}{4}\cdot\frac{1}{1-4x} + C =$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} (4x)^n + C = \sum_{n=0}^{\infty} 4^{n-1} x^n + C$$

where $|4x| < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$

We have $\dfrac{d}{dx}\displaystyle\int g(x)\,dx = \dfrac{d}{dx}\left(\sum_{n=0}^{\infty} 4^{n-1} x^n + C\right)$

$$g(x) = \sum_{n=1}^{\infty} 4^{n-1} n\, x^{n-1}$$

$$f(x) = x^4 g(x) = x^4 \sum_{n=1}^{\infty} 4^{n-1} n\, x^{n-1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} 4^{n-1} n\, x^{n+3}}$$

$$\boxed{\left(-\frac{1}{4}, \frac{1}{4}\right)}$$

Test end point for $g(x)$

$x = -\frac{1}{4}$

$g(x) = \sum_{n=1}^{\infty} 4^{n-1} n \cdot \left(-\frac{1}{4}\right)^{n-1} = \underbrace{\sum_{n=1}^{\infty} (-1)^{n-1} n}$ divergent by DT

$x = \frac{1}{4}$

$g(x) = \sum_{n=1}^{\infty} 4^{n-1} n \left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{\infty} n$ divergent by DT

(h) $f(x) = \arctan(16x^4)$

$$f'(x) = \frac{64x^3}{1+(16x^4)^2} = 64x^3\cdot\frac{1}{1+16^2\cdot x^8} =$$

$$= 64x^3\,\frac{1}{1-(-16^2x^8)} = 64x^3\sum_{n=0}^{\infty}(-16^2x^8)^n$$

$$f'(x) = \sum_{n=0}^{\infty} 64\cdot(-1)^n\, 16^{2n}\, x^{8n+3} \quad \text{, where } |16^2x^8|<1$$

$$|(2x)^8|<1$$

$$|2x|<1$$

$$|x|<\tfrac{1}{2}$$

$$-\tfrac{1}{2}<x<\tfrac{1}{2}$$

$$f(x)=\int f'(x)\,dx = \sum_{n=0}^{\infty} 2^6(-1)^n(2^4)^{2n}\int (x^{8n+3})\,dx$$

$$f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\,\frac{x^{8n+4}}{8n+4} + C$$

If $x=0 \Rightarrow \arctan(16\cdot 0^4) = 0 = f(0) = 0 + C \Rightarrow C = 0$

$$\arctan(16x^4) = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\,\frac{x^{8n+4}}{8n+4}$$

Test end points $x = \pm\frac{1}{2}$

$x = -\frac{1}{2}$

$$f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\,\frac{\left(-\frac{1}{2}\right)^{8n+4}}{8n+4} = \sum_{n=0}^{\infty}\frac{(-1)^n\,\dfrac{2^{8n+6}}{2^{8n+4}}\cdot\underbrace{(-1)^{\overset{\text{even}}{8n+4}}}_{=1}}{8n+4}$$

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^2}{8n+4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \quad \text{conv. by AST}$$

$x=\frac{1}{2}$ $\quad f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{8n+6}\,\frac{\left(\frac{1}{2}\right)^{8n+4}}{8n+4}$ // conv. by AST (see above)

$$\boxed{-\tfrac{1}{2} \le x \le \tfrac{1}{2}}$$