6.1: Sigma notation

Σ

DEFINITION 1. *If $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ are real numbers and m and n are integers such that $m \le n$, then*

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n.$$

a counter $i = m, m+1, m+2, \dots, n$

$$(-1)^k = \begin{cases} 1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}$$

EXAMPLE 2. *Compute the summation*

$$\sum_{k=1}^{4} \frac{(-1)^k}{k} = \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4}$$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \ldots$$

EXAMPLE 3. *Write the sum in sigma notation:*

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} = \sum_{k=1}^{5} \frac{1}{k^2}$$

THEOREM 4. *If c is any constant then*

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Note that in general

$$\sum_{i=m}^{n} a_i b_i \neq \left(\sum_{i=m}^{n} a_i\right) \cdot \left(\sum_{i=m}^{n} b_i\right).$$

$$\sum_{k=3}^{7} \frac{5k}{k+1} = 5\sum_{k=3}^{7}\frac{k}{k+1}$$

$$\sum_{j=m}^{\ell}(3a_j - 9b_j) = 3\sum_{j=m}^{\ell} a_j - 9\sum_{j=m}^{\ell} b_j$$

THEOREM 5.

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} c = nc$, *where c is a constant.*
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 1+2+3+\dots+n$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = 1^2+2^2+3^2+\dots+n^2$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = 1^3+2^3+3^3+\dots+n^3$

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\dots+1}_{n \text{ times}} = n$$

$$\sum_{i=1}^{n} 77 = 77\sum_{i=1}^{n} 1 = 77n$$

EXAMPLE 6. *Compute these sums:*

(a) $\sum_{i=1}^{n} i(i+4) = \sum_{i=1}^{n} (i^2+4i) = \sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i \overset{\text{Th.5}}{=}$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{6} + 2n(n+1)$$

(b) $\sum_{j=1}^{n} \left[\left(\frac{j}{n}\right)^3 + 1\right] = \sum_{j=1}^{n} \frac{j^3}{n^3} + \sum_{j=1}^{n} 1$

$$= \frac{1}{n^3} \sum_{j=1}^{n} j^3 + \sum_{j=1}^{n} 1 \overset{\text{th.5}}{=}$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)}{2}\right]^2 + n$$

$$= \frac{\cancel{n^2}(n+1)^2}{4n^{\cancel{3}}} + n = \frac{(n+1)^2 + 4n^2}{4n} = \frac{n^2+2n+1+4n^2}{4n}$$

$$= \frac{5n^2+2n+1}{4n}$$

EXAMPLE 7. *Find the limit:* $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n}\left[\left(\frac{j}{n}\right)^3 + 1\right] =$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{j=1}^{n}\left[\left(\frac{j}{n}\right)^3 + 1\right]\right) \overset{\text{Ex. 6(b)}}{=}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{5n^2+2n+1}{4n}$$

$$= \lim_{n\to\infty} \frac{5n^2+2n+1}{4n^2} = \boxed{\frac{5}{4}}$$

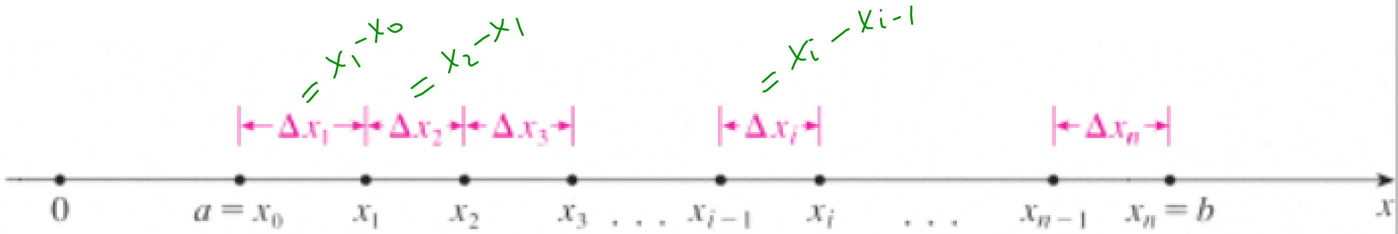
6.2: Area

Area problem: Let a function $f(x)$ be positive on some interval $[a, b]$. Determine the area of the region S between the function and the x-axis.

Solution: Choose **partition** points $x_0, x_2, \ldots, x_{n-1}, x_n$ so that

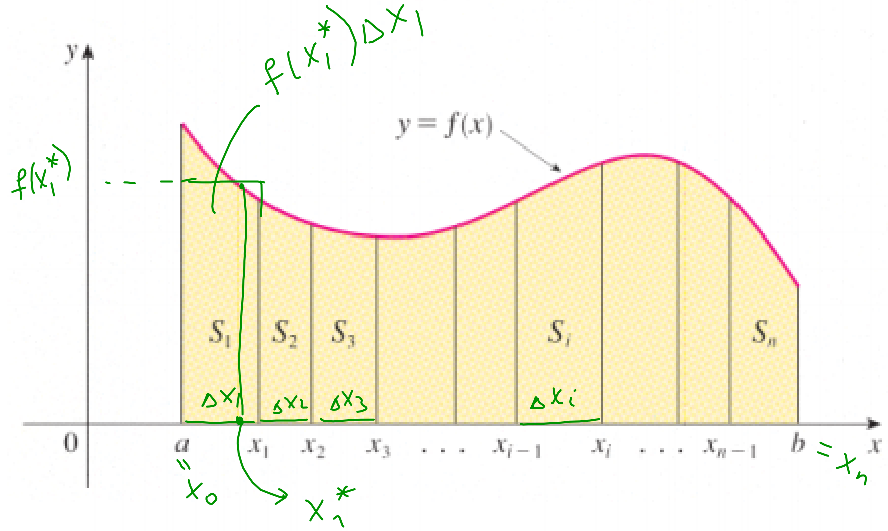
$$a = x_0 \le x_2 \le \ldots \le x_{n-1} \le x_n = b.$$

Use notation $\Delta x_i = x_i - x_{i-1}$ for the length of ith subinterval $[x_{i-1}, x_i]$ $(1 \le 1 \le n)$



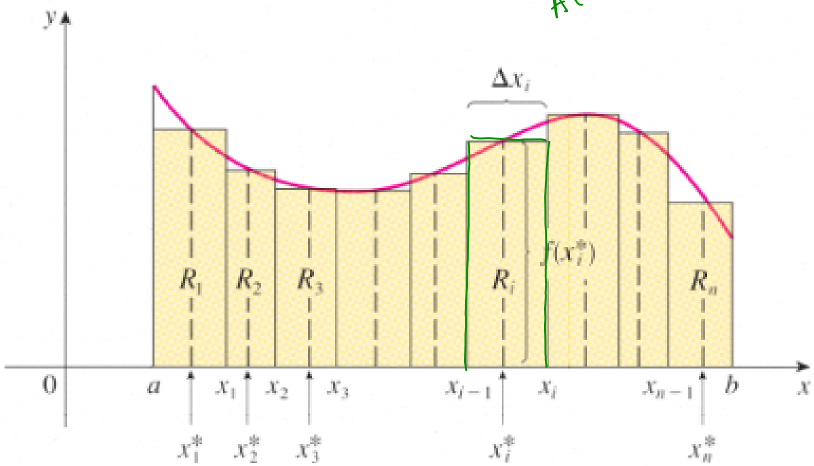
The length of the longest subinterval is denoted by $\|P\|$ norm of partition $= \max_i |\Delta x_i|$

Use the partition P to divide the region S into strips $S_1, S_2, \ldots, S_n$.



Approximate the strips $S_1, S_2, \ldots, S_n$ by rectangles $R_1, R_2, \ldots, R_n$.

$A(R_i) = \Delta x_i \cdot f(x_i^*)$



The location in each subinterval where we compute the height is denoted by x_i^*.

The area of the ith rectangle is

$$A_i = f(x_i^*)\,\Delta x_i$$

Then

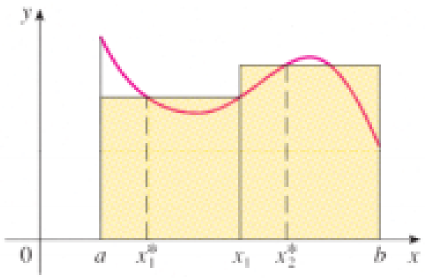
$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The area A of the region is:

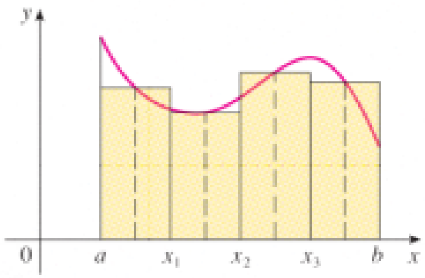
$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Riemann Sum

$\|P\| = \max |\Delta x_i|$

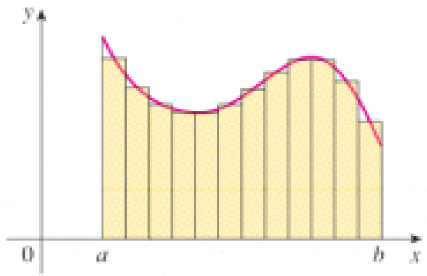


(a) $n = 2$



(b) $n = 4$

(c) $n = 8$



(d) $n = 12$

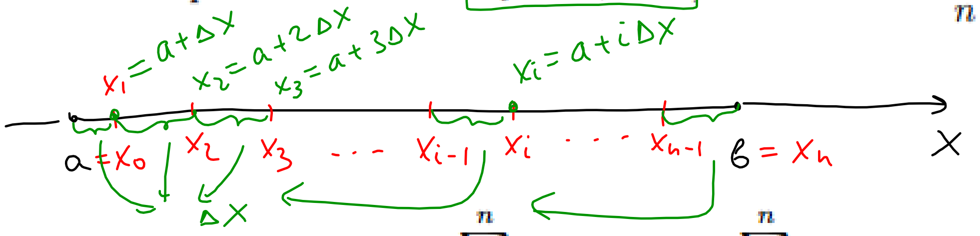
Riemann Sum for a function $f(x)$ on the interval $[a, b]$ is a sum of the form:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

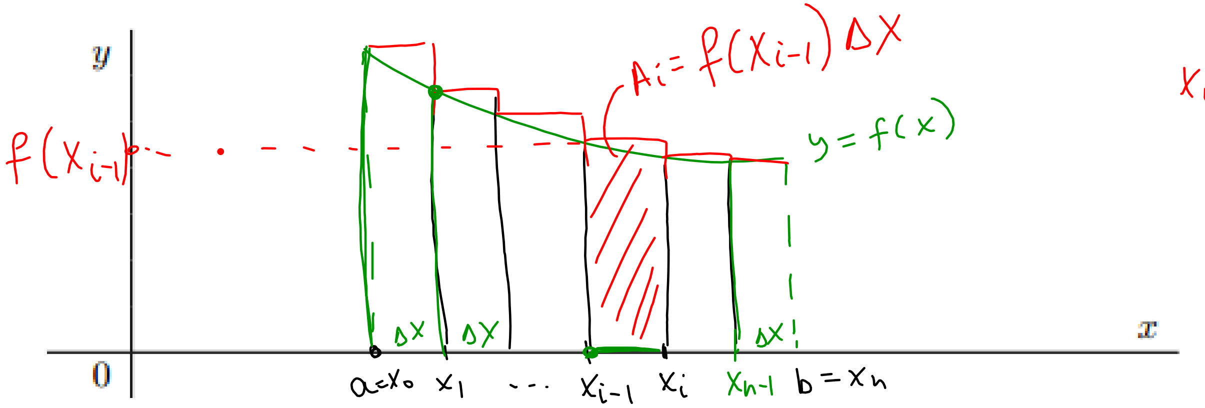
Consider a partition has equal subintervals: $\boxed{x_i = a + i\Delta x}$ where $\Delta x = \frac{b-a}{n}$.

length of interval $[a,b]$

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$



LEFT-HAND RIEMANN SUM : $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(\underbrace{a + (i-1)\Delta x}_{})\Delta x$



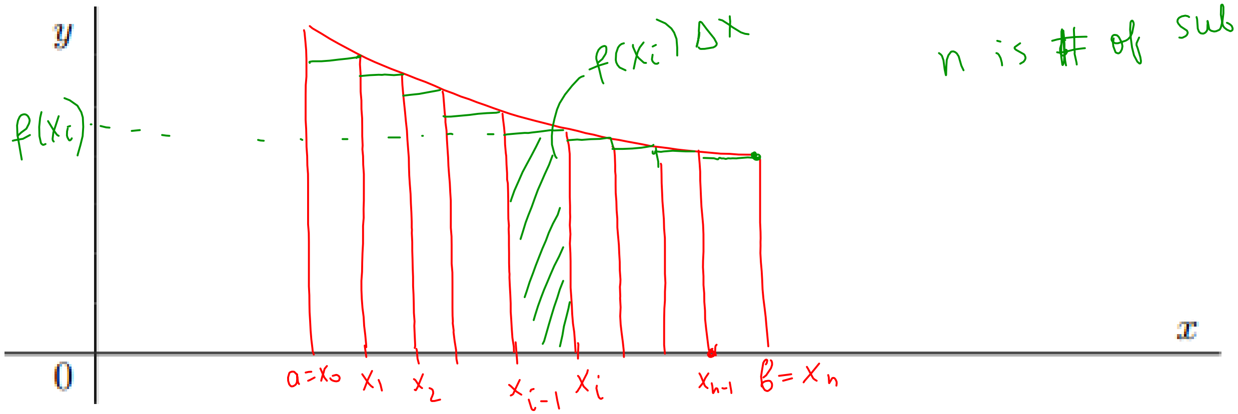
$x_{i-1} = a + (i-1)\Delta x$

$\Delta x = \frac{b-a}{n}$

$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$

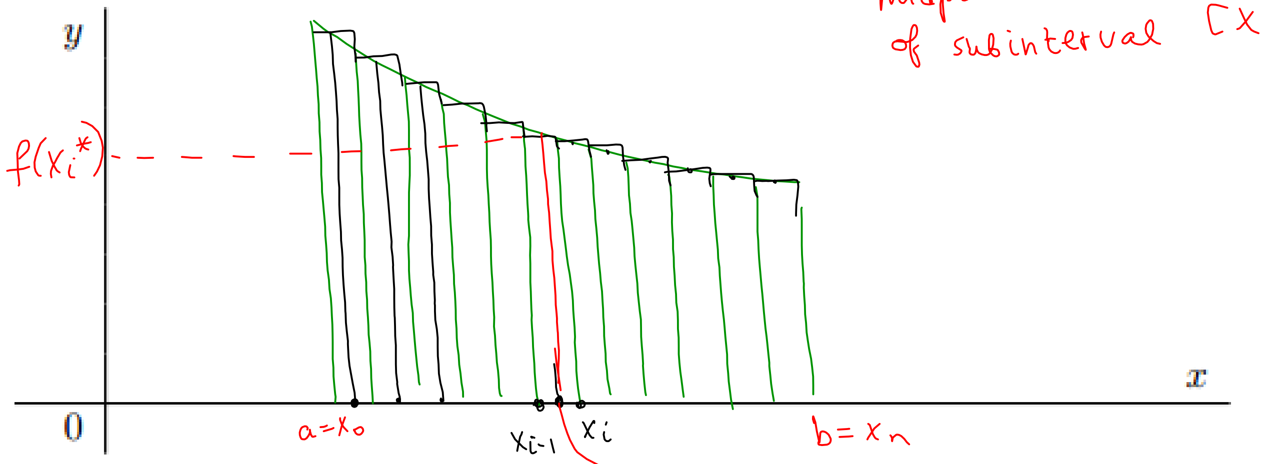
RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(\underbrace{a+i\Delta x}_{x_i})\Delta x$

n is # of subintervals



MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\underbrace{\frac{x_i + x_{i-1}}{2}}_{\text{midpoint}}\right)\Delta x$

midpoint of subinterval $[x_{i-1}, x_i]$

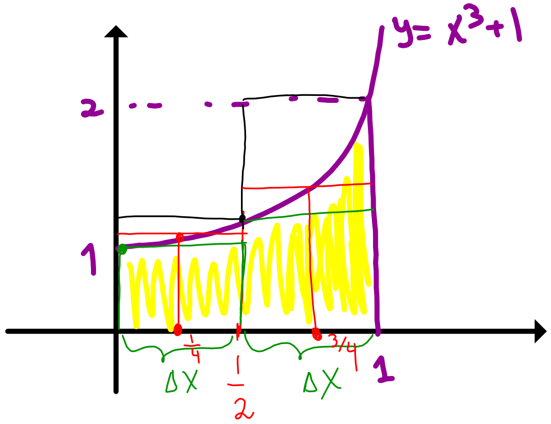


$x_i^* = \dfrac{x_i + x_{i-1}}{2}$

EXAMPLE 8. *Given* $f(x) = x^3 + 1$ *on* $[0, 1]$.

(a) *Calculate* L_2, R_2, M_2.

$\boxed{n=2}$ $\quad \Delta x = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}$



$$L_2 = f(0)\Delta x + f\left(\tfrac{1}{2}\right)\Delta x = \left(f(0) + f\left(\tfrac{1}{2}\right)\right)\Delta x = \left(1 + \left(\left(\tfrac{1}{2}\right)^3 + 1\right)\right)\cdot\tfrac{1}{2}$$

$$= \left(2 + \tfrac{1}{8}\right)\cdot\tfrac{1}{2} = 1 + \tfrac{1}{16} = \tfrac{17}{16}$$

$$R_2 = \left(f\left(\tfrac{1}{2}\right) + f(1)\right)\Delta x = \left(\left(\tfrac{1}{2}\right)^3 + 1 + 1^3 + 1\right)\cdot\tfrac{1}{2}$$

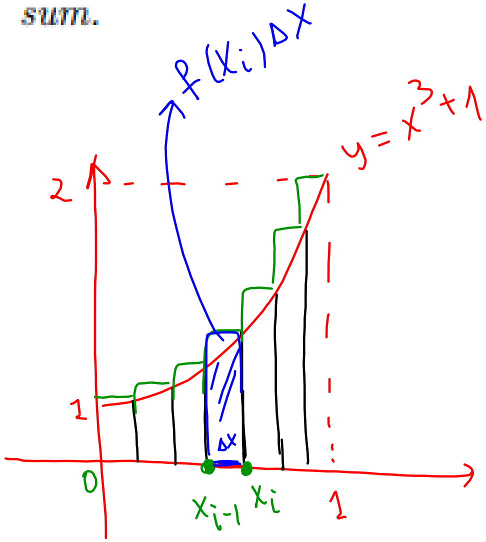
$$= \left(\tfrac{1}{8} + 3\right)\tfrac{1}{2} = \tfrac{25}{8}\cdot\tfrac{1}{2} = \tfrac{25}{16}$$

$$M_2 = \left(f(\underbrace{\tfrac{1}{4}}_{\frac{0+\frac{1}{2}}{2}}) + f(\underbrace{\tfrac{3}{4}}_{\frac{\frac{1}{2}+1}{2}})\right)\Delta x = \left(\left(\tfrac{1}{4}\right)^3 + 1 + \left(\tfrac{3}{4}\right)^3 + 1\right)\tfrac{1}{2}$$

$$= \left(\tfrac{1}{64} + \tfrac{27}{64} + 2\right)\tfrac{1}{2} = \left(\tfrac{28}{64} + 2\right)\cdot\tfrac{1}{2}$$

$$= \left(\tfrac{7}{16} + 2\right)\tfrac{1}{2} = \tfrac{7}{32} + 1 = \tfrac{39}{32}$$

(b) *Represent area bounded by* $f(x) = x^3 + 1$ *on the interval* $[0, 1]$ *using right endpoints by Riemann sum.*



$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$$

$$x_i = \underbrace{a}_{=0} + i\,\Delta x = i\cdot\frac{1}{n} = \frac{i}{n}$$

$$= \sum_{i=1}^{n}\left(\left(\frac{i}{n}\right)^3 + 1\right)\frac{1}{n}$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{i}{n}\right)^3 + 1\right)\frac{1}{n}$$

(c) *Find the area bounded by* $f(x) = x^3 + 1$ *on the interval* $[0, 1]$.

By Ex. 7 $\quad A = \frac{5}{4}$

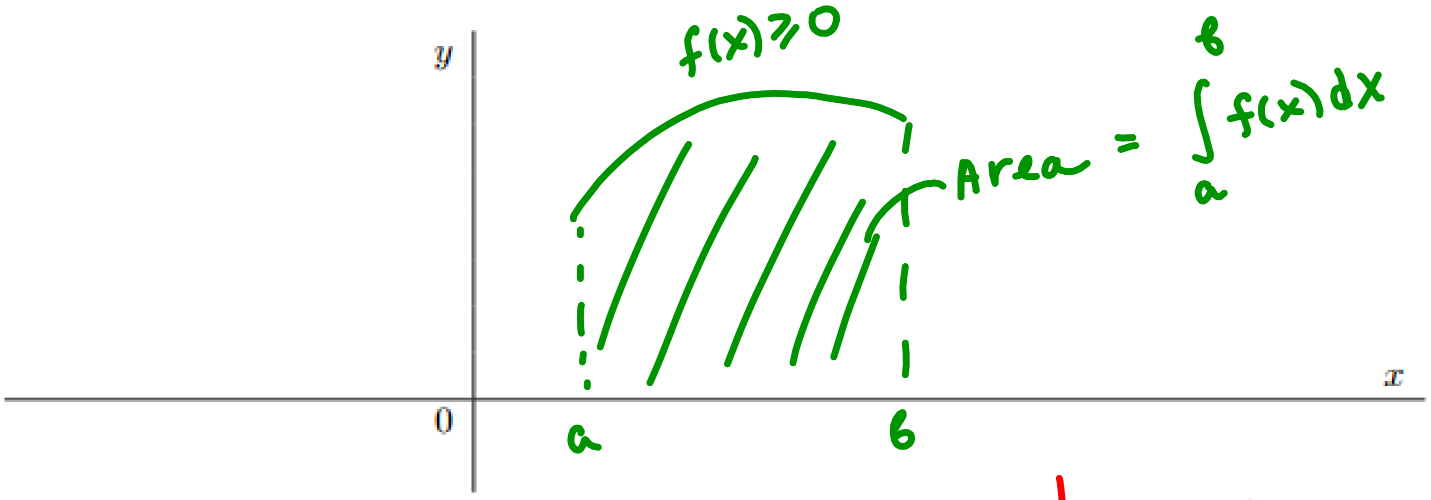
6.3: The Definite Integral

DEFINITION 9. *The* **definite integral of** f **from** a **to** b *is*

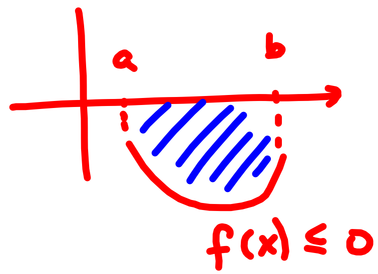
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f *is called* **integrable** *on the interval* $[a, b]$.

If $f(x) > 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function f and the lines $y = 0$, $x = a$ and $x = b$.



$$\int_a^b f(x)\,dx = -A$$



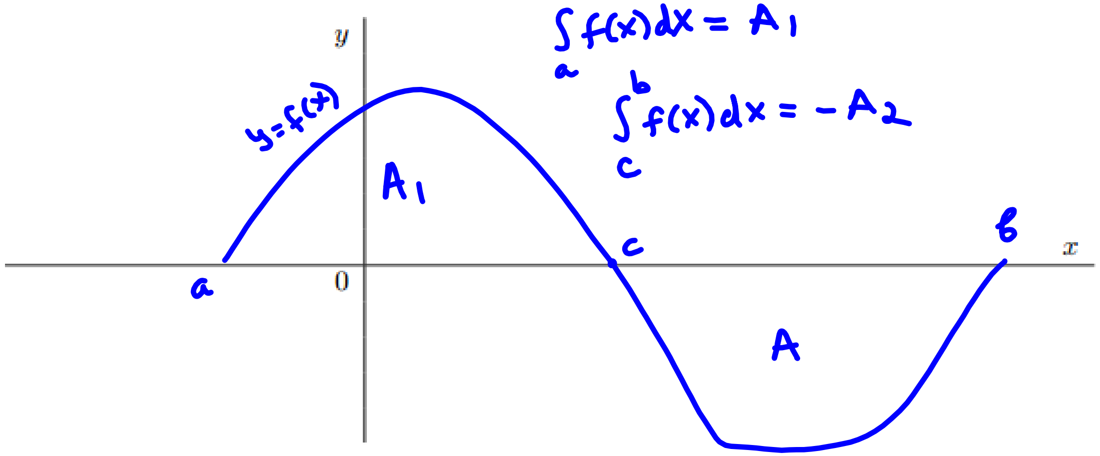
In general, a definite integral can be interpreted as a difference of areas:

$$\boxed{\int_a^b f(x)\,dx = A_1 - A_2}$$

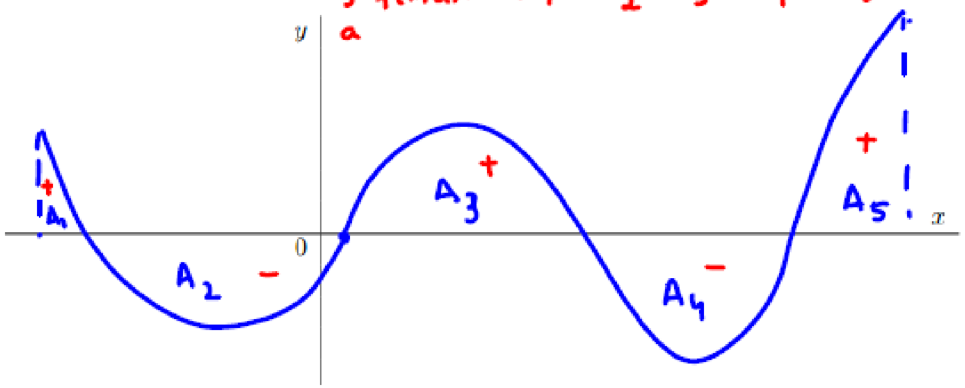
where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the regi
below the x and above the graph of f.

$\int_a^c f(x)dx = A_1$

$\int_c^b f(x)dx = -A_2$

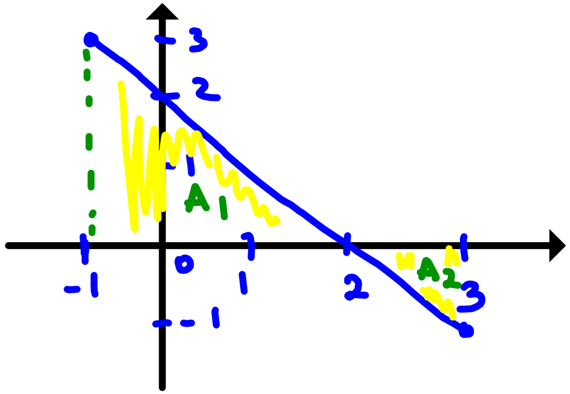


$\int_a^b f(x)dx = A_1 - A_2 + A_3 - A_4 + A_5$



EXAMPLE 10. *Evaluate the integrals by* $\int_{-1}^{3}(2-x)\,dx$ *interpreting it in terms of areas.*

Graph $y=f(x)=2-x$



$$\int_{-1}^{3}(2-x)\,dx = A_1 - A_2 = \frac{3\cdot 3}{2} - \frac{1\cdot 1}{2} = 4$$

Properties of Definite Integrals:

- $\int_a^b dx = b - a$
- $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
- $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$, where c is any constant
- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, where $a \leq c \leq b$
- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
- $\int_a^a f(x)\, dx = 0$
- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\, dx$

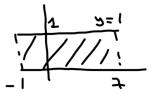
EXAMPLE 11. *Write as a single integral:*

$$\int_3^5 f(x)\, dx + \int_0^3 f(x)\, dx - \int_6^5 f(x)\, dx + \int_5^5 f(x)\, dx = \int_{\square}^{\square} f(x)dx$$

$$= \int_3^5 f(x)dx + \int_0^3 f(x)dx + \int_5^6 f(x)dx + 0 = \int_0^3 + \int_3^5 + \int_5^6$$

$$= \int_0^6 f(x)\, dx.$$

$$\int_{-1}^{7} dx = \int_{-1}^{7} 1\cdot dx = 1\cdot(7-(-1)) = 8$$



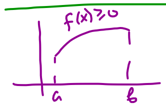
$$\int_{7}^{5\pi} \left(3\tan x - 5e^{x^3}\right)dx =$$

$$= 3\int_{7}^{5\pi} \tan x\,dx - 5\int_{7}^{5\pi} e^{x^3}\,dx$$

$$\int_{-5}^{16} x^3\,dx = \int_{-5}^{-1} x^3\,dx + \int_{-1}^{10} x^3\,dx + \int_{10}^{16} x^3\,dx$$

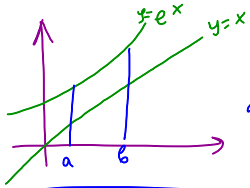
$$\int_{5}^{8} f(x)\,dx = -\int_{8}^{5} f(x)\,dx$$

$$\int_{100}^{100} e^x \ln(\sin^2 x)\,dx = 0$$



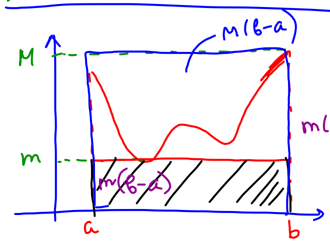
$$\int_{-15}^{-10} (e^x + 33)\,dx \geq 0$$

$$\int_{-1}^{5} x^2\,dx \geq 0$$



$e^x > x$

$$\int_a^b e^x\,dx > \int_a^b x\,dx$$

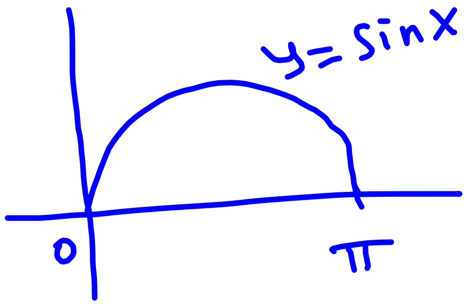


on $[a,b]$

$m \leq f(x) \leq M$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

EXAMPLE 12. *Estimate the value of* $\int_0^{\pi} (4\sin^5 x + 3)\,dx$



on $[0, \pi]$

$$0 \le \sin x \le 1$$

$$0 \le \sin^5 x \le 1$$

$$0 \le 4\sin^5 x \le 4$$

$$m = 3 \le 4\sin^5 x + 3 \le 7 = M$$

$$b - a = \pi - 0 = \pi$$

$$3\pi \le \int_0^{\pi} (4\sin^5 x + 3)\,dx \le 7\pi$$

6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

EXAMPLE 13. *Differentiate* $g(x) = \int_{-4}^{x} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t)}\,dt$

$$g'(x) = f(x) = e^{2x}\cos^2(1-5x)$$

Remark: If $g(x) = \int_x^a f(t)\,dt$

then $g'(x) = -f(x)$,

because $\int_x^a = -\int_a^x$

$$\frac{d}{dx}\left(\int_x^a f(t)\,dt\right) = -f(x).$$

THEOREM 14. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x).$$

EXAMPLE 15. *Differentiate* $g(x) = \int_{-4}^{x^3} e^{2t}\cos^2(1-5t)\,\mathrm{d}t$.

$t \leftrightarrow x^3$

$$g'(x) = \left(e^{2x^3}\cos^2(1-5x^3)\right)\cdot \underbrace{3x^2}_{=(x^3)'}$$

Remark: $\frac{d}{dx}\left(\int_{v(x)}^{u(x)} f(t)\,dt\right) = f(u(x))u'(x) - f(v(x))v'(x)$

Example $\left(\int_x^{e^x} \sin^2 t\,dt\right)' = \sin^2(e^x)\cdot(e^x)' - \sin^2 x\,(x')$

$= e^x \sin^2 e^x - \sin^2 x$

$$F'(x) = f(x)$$

PART II If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

$$\int_0^1 2x\,dx = (x^2 + c)\Big|_0^1 = (1^2 + \cancel{c}) - (0^2 + \cancel{c}) = 1.$$

EXAMPLE 16. *Evaluate*

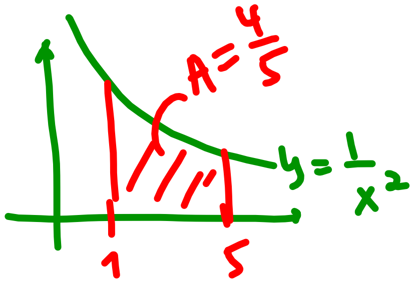
(a) $\displaystyle\int_{-\pi/2}^{0} (\cos x - 4\sin x)\,\mathrm{d}x = \left(\sin x - 4(-\cos x)\right)\Big|_{-\pi/2}^{0}$

$$= (\sin x + 4\cos x)\Big|_{-\pi/2}^{0}$$

$$= \sin 0 + 4\cos 0 - \left(\sin\left(\tfrac{-\pi}{2}\right) + 4\cos\left(\tfrac{-\pi}{2}\right)\right)$$

$$= 0 + 4 - (-1 + 4\cdot 0) = 5.$$

(b) $\int_1^5 \frac{1}{x^2}\,dx = \int_1^5 x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^5 = -\left(\frac{1}{x}\right)\Big|_1^5$



$$= -\left(\frac{1}{5} - \frac{1}{1}\right) = -\left(-\frac{4}{5}\right) = \frac{4}{5}$$

(c) $\int_0^1 (u^3+2)^2\,du = \int_0^1 \left(u^6 + 4u^3 + 4\right)du = \left(\frac{u^{6+1}}{6+1} + \frac{4u^{3+1}}{3+1} + 4u\right)\Big|_0^1$

$$= \left(\frac{u^7}{7} + u^4 + 4u\right)\Big|_0^1 = \frac{1}{7} + 1 + 4 - 0$$

$$= \frac{36}{7}$$

Indefinite integral

$$\int f(x)\,dx = F(x) + C$$

the most general antiderivative

$$F'(x) = f(x)$$

For example

$$\int e^x\,dx = e^x + C$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{2x\sqrt{x}}{3} + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$