Section 1.3: Vector functions and parametric curves

Parametric equations:

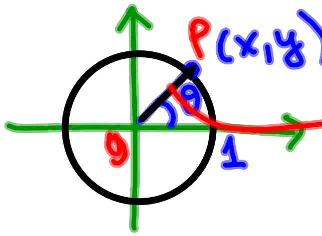
$$\boxed{x = x(t), \quad y = y(t)}$$

where the variable t is called a **parameter.** Each value of the parameter t defines a point that we can plot. As t varies over its domain we get a collection of points $(x, y) = (x(t), y(t))$ on the plane which is called the **parametric curve.**

Each parametric curve can be represented as the **vector function:**

$$\boxed{\overrightarrow{r(t)} = \langle x(t), y(t) \rangle \, .}$$

Note that Parametric curves have a **direction of motion** given by increasing of parameter t. So, when sketching parametric curves we also include arrows that show the direction of motion.



$P(x, y) = (\cos\theta, \sin\theta)$

$\langle \cos\theta, \sin\theta \rangle = \overrightarrow{OP}$

$$\boxed{x = \cos t, \quad y = \sin t}$$

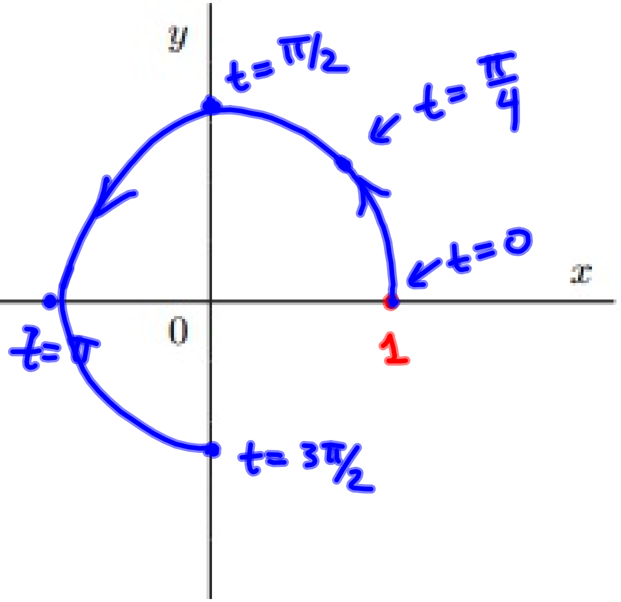
$$\boxed{\langle \cos t, \sin t \rangle = \vec{r}(t)}$$

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2$$

$$\boxed{x^2 + y^2 = 1}$$

EXAMPLE 1. **(a)** *Examine the parametric curve* $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 3\pi/2.$

t	0	$\pi/4$	$\pi/2$	π	$\frac{3\pi}{2}$
(x, y)	$(1,0)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$(0,1)$	$(-1,0)$	$(0,-1)$



(b) *Find the Cartesian equation for* $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

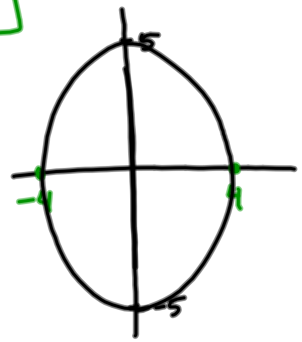
We need to eliminate the parameter t.
Use the following trig. identity for that:

$$\cos^2 t + \sin^2 t = 1$$

$$(\underbrace{\cos t}_{x})^2 + (\underbrace{\sin t}_{y})^2 = 1$$

$$\boxed{x^2 + y^2 = 1}$$

(c) *Find parametric equation of the curve* $\frac{x^2}{16} + \frac{y^2}{25} = 1$.



$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$

$$\Big(\underbrace{\frac{x}{4}}_{X}\Big)^2 + \Big(\underbrace{\frac{y}{5}}_{Y}\Big)^2 = 1$$

$$X^2 + Y^2 = 1$$

$$X = \frac{x}{4} = \cos t \;, \qquad Y = \frac{y}{5} = \sin t$$

$$\boxed{x = 4\cos t, \quad y = 5\sin t}$$

Remark

In general, parametric equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are

$$x = a\cos t \;, \quad y = b\sin t$$

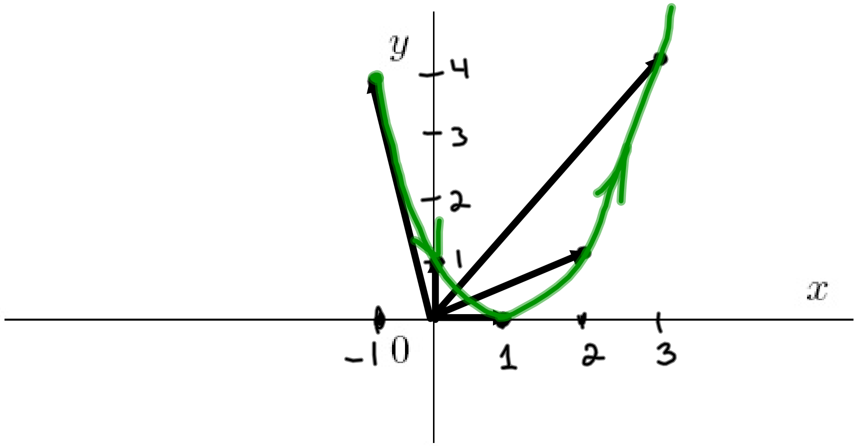
EXAMPLE 2. *Given* $\mathbf{r}(t) = \langle t+1, t^2 \rangle$. or $\boxed{x = t+1,\ y = t^2}$

(a) *Does the point* $(4, 3)$ *belong to the graph of* $\mathbf{r}(t)$? NO

$\vec{r}(t) = \langle 4,3 \rangle \Longrightarrow \langle t+1, t^2 \rangle = \langle 4,3 \rangle \Longrightarrow \begin{cases} t+1=4 \\ t^2=3 \end{cases} \Longrightarrow \begin{cases} t=3 \\ t=\pm\sqrt{3} \end{cases}$ a contradiction

(b) *Sketch the graph of* $\mathbf{r}(t)$.

t	$\mathbf{r}(t)$
-2	$\langle 1,4 \rangle$
-1	$\langle 0,1 \rangle$
0	$\langle 1,0 \rangle$
1	$\langle 2,1 \rangle$
2	$\langle 3,4 \rangle$



(c) *Find the Cartesian equation of* $\mathbf{r}(t)$ *eliminating the parameter.*

$\begin{cases} x = t+1 \Longrightarrow t = x-1 \\ y = t^2 \end{cases}$

$\boxed{y = (x-1)^2}$ parabola

EXAMPLE 3. *Find the Cartesian equation for* $\mathbf{r}(t) = \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\cos(2t)}_{y}\,\mathbf{j}$

$$\begin{cases} x = \cos t \\ y = \cos(2t) \end{cases}$$

To find the Cartesian equation for $\vec{r}(t)$ we eliminate the parameter t using the following trig. identity:

$$\cos 2t = 2\cos^2 t - 1$$

$$\underbrace{\cos 2t}_{y} = 2(\underbrace{\cos t}_{x})^2 - 1$$

$$\boxed{y = 2x^2 - 1}$$ parabola

EXAMPLE 4. *An object is moving in the xy-plane and its position after t seconds is given by* $\mathbf{r}(t) = \langle 1+t^2, 1+3t \rangle$. or $x = 1+t^2$, $y = 1+3t$.

(a) *Find the position of the object at time* $t = 0$s

$$\vec{r}(0) = \langle 1+0^2, 1+3\cdot 0 \rangle = \boxed{\langle 1, 1 \rangle}$$

(b) *At what time does the object reach the point* $(10, 10)$.

Find t such that $\vec{r}(t) = \langle 10, 10 \rangle$

$$\langle 1+t^2, 1+3t \rangle = \langle 10, 10 \rangle$$

$$\left\{\begin{array}{l} 1+t^2 = 10 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3 \\ 1+3t = 10 \Rightarrow 3t = 9 \Rightarrow t = 3 \end{array}\right\} \Rightarrow \boxed{t=3}$$

Answer: $t = 3$s

(c) *Does the object pass through the point* $(20, 20)$? $\boxed{\text{NO}}$

$$\left\{\begin{array}{l} 1+t^2 = 20 \\ 1+3t = 20 \Rightarrow 3t = 19 \Rightarrow \boxed{t = \frac{19}{3}} \end{array}\right.$$

$$1 + \left(\frac{19}{3}\right)^2 = 20 \text{ (a contradiction)}$$

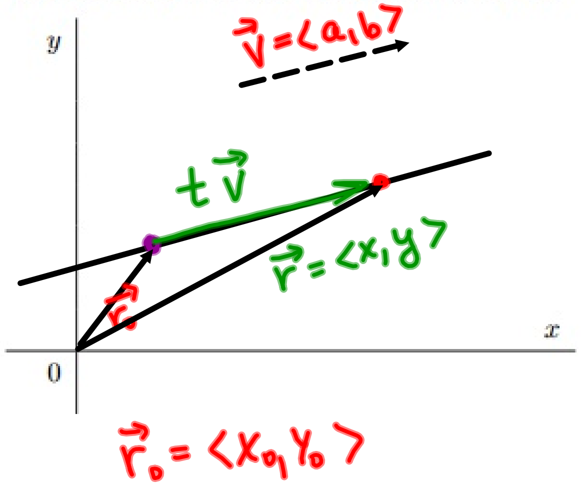
(d) *Find an equation in x and y whose graph is the path of the object.*

Find the Cartesian equation for $\vec{r}(t)$ eliminating the parameter t:

$$\left\{\begin{array}{l} x = 1+t^2 \\ y = 1+3t \Rightarrow 3t = y-1 \Rightarrow \boxed{t = \frac{y-1}{3}} \end{array}\right.$$

$$\boxed{x = 1 + \left(\frac{y-1}{3}\right)^2}$$

Vector and parametric equations of line



By Triangle law:

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

Line through the point $\underbrace{(2,9)}_{=\vec{r}_0}$ and parallel to vector $\underbrace{\langle 7, 9 \rangle}_{=\vec{v}}$

$$\boxed{\vec{r}(t) = \langle 2, 9 \rangle + t\langle 7, 9 \rangle}$$

$$\langle x, y \rangle = \langle 2 + 7t,\ 9 + 9t \rangle$$

$$\boxed{x = 2 + 7t, \quad y = 9 + 9t}$$

A **Vector equation of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b \rangle$ is given by

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v},$$

where $\mathbf{r_0} = \langle x_0, y_0 \rangle$.

The vector equation of the line can be rewritten in parametric form. Namely, we have

$$\begin{aligned} \langle x(t), y(t) \rangle &= \mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \\ &= \langle x_0, y_0 \rangle + t\,\langle a, b \rangle = \langle x_0, y_0 \rangle + \langle ta, tb \rangle = \\ &= \langle x_0 + ta, y_0 + tb \rangle . \end{aligned}$$

coordinates of a point on the line

$$x = x_0 + t\,a$$
$$y = y_0 + t\,b$$

components of the vector parallel to the line

This immediately yields that the **parametric equations of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b \rangle$ are

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt.$$

EXAMPLE 5. *Find parametric equations of the line*

(a) *passing through the point* $\underbrace{(1,0)}_{=(x_0,y_0)}$ *and parallel to the vector* $\underbrace{\mathbf{i}-4\mathbf{j}}_{=\vec{v}=\langle 1,-4\rangle}$;

$$x = 1 + 1\cdot t$$
$$y = 0 + (-4)t$$

or

$$\boxed{\begin{aligned} x &= 1+t \\ y &= -4t \end{aligned}}$$

$$\boxed{\begin{aligned} x &= 1+2t \\ y &= -8t \end{aligned}}$$

another possible answer

$$\boxed{\begin{aligned} x &= 2+t \\ y &= -4-4t \end{aligned}}$$

(b) *passing through the point* $\underbrace{(-4,5)}_{=(x_0,y_0)}$ *with slope* $\sqrt{3}$;

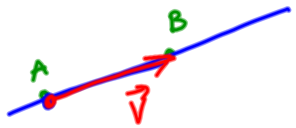
$$\vec{v} = \langle 1, \sqrt{3}\rangle$$

$$x = -4 + 1\cdot t$$
$$y = 5 + \sqrt{3}\,t$$

OR

$$\boxed{\begin{aligned} x &= -4+t \\ y &= 5+\sqrt{3}\,t \end{aligned}}$$

(c) *passing through the points* $\overset{A}{(7,2)}$ *and* $\overset{B}{(3,2)}$.



$$\vec{AB} = \langle 3-7, 2-2\rangle = \langle -4, 0\rangle = -4\vec{i}$$

In particular, choose $\vec{v} = \vec{i}$

$$x = 7 + 1\cdot t$$
$$y = 2 + 0\cdot t$$

OR

$$\boxed{\begin{aligned} x &= 7+t \\ y &= 2 \end{aligned}}$$

Some other possible answers:

$$\boxed{\begin{aligned} x &= 3+t \\ y &= 2 \end{aligned}} \qquad \boxed{\begin{aligned} x &= 3+\tfrac{1}{2}t \\ y &= 2 \end{aligned}} \qquad \boxed{\begin{aligned} x &= 7+2017t \\ y &= 2 \end{aligned}}$$

$\vec{V}_1 = \langle 1, -3 \rangle$ $\qquad$ $\vec{V}_2 = \langle 3, 1 \rangle$

EXAMPLE 6. *Determine whether the lines* $\mathbf{r}(t) = \langle 1+t, 1-3t \rangle$, $\mathbf{R}(s) = \langle 1+3s, 12+s \rangle$ *are parallel, orthogonal or neither. If they are not parallel, find the intersection point.*

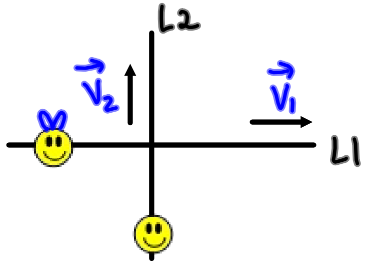
L1 : $\vec{r}(t) = \langle 1+t, 1-3t \rangle$ with direction $\vec{V}_1 = \langle 1, -3 \rangle$

L2 : $\vec{R}(s) = \langle 1+3s, 12+s \rangle$ with direction $\vec{V}_2 = \langle 3, 1 \rangle$

$\frac{1}{3} \neq \frac{-3}{1} \Rightarrow \vec{V}_1 \nparallel \vec{V}_2 \Rightarrow L1 \nparallel L2$ (non parallel)

$\vec{V}_1 \cdot \vec{V}_2 = \langle 1, -3 \rangle \cdot \langle 3, 1 \rangle = 1 \cdot 3 + (-3) \cdot 1 = 0 \Rightarrow \vec{V}_1 \perp \vec{V}_2 \Rightarrow$

$L1 \perp L2$ (orthogonal).



To find the intersection point, first find s and t such that

$$\vec{r}(t) = \vec{R}(s):$$

$$\langle 1+t, 1-3t \rangle = \langle 1+3s, 12+s \rangle$$

$$\begin{cases} 1+t = 1+3s \Rightarrow t = 3s \\ 1-3t = 12+s \end{cases}$$

$$1-3(3s) = 12+s$$

$$1-9s = 12+s$$

$$-9s-s = 12-1$$

$$-10s = 11 \Rightarrow \boxed{s = -\frac{11}{10}}$$

$$t = 3s = 3\cdot\left(-\frac{11}{10}\right) = \boxed{-\frac{33}{10} = t}$$

$$\vec{R}\left(-\frac{11}{10}\right) = \left\langle 1+3\left(-\frac{11}{10}\right), 12+\left(-\frac{11}{10}\right) \right\rangle$$

$$= \left\langle 1-\frac{33}{10}, 12-\frac{11}{10} \right\rangle = \left\langle \frac{10-33}{10}, \frac{120-11}{10} \right\rangle$$

$$\vec{R}\left(-\frac{11}{10}\right) = \left\langle -\frac{23}{10}, \frac{109}{10} \right\rangle \Rightarrow$$ the intersection point is $\boxed{\left(-\frac{23}{10}, \frac{109}{10}\right)}$

Remark: BTW, $\vec{r}\left(-\frac{33}{10}\right) = \vec{R}\left(-\frac{11}{10}\right)$.