Section 2.3: Calculating limits using the limits laws

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

$$\lim_{x\to a}(\sin x + \cos x) = \lim_{x\to a}\sin x + \lim_{x\to a}\cos x$$

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$

$$0 = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) \neq \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x} \quad \text{DNE}$$

$$\lim_{x\to a} 5\sin x = 5\lim_{x\to a}\sin x$$

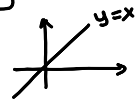
$$\lim_{x\to a}(c_1 f(x) \pm c_2 g(x)) = c_1\lim_{x\to a} f(x) \pm c_2 \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(\sin x \cos x) = \left(\lim_{x\to a}\sin x\right)\lim_{x\to a}\cos x$$

$$1 = \lim_{x\to 0}\left(\frac{1}{x}\cdot x\right) \neq \underbrace{\lim_{x\to 0}\frac{1}{x}}_{\text{DNE}} \cdot \underbrace{\lim_{x\to 0} x}_{=0} \quad \text{DNE}$$

$$1 = \lim_{x\to 0}\frac{x}{x} \neq \frac{\lim_{x\to 0} x}{\lim_{x\to 0} x} \; ?$$

6. $\lim_{x\to a} x = a$



7. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer.
8. $\lim_{x\to a} x^n = a^n$, where n is a positive integer.
9. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer and if n is even, then we assume that $\lim_{x\to a} f(x) > 0$.
10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}$ where n is a positive integer and if n is even, then we assume that $a > 0$.

$$\lim_{x\to a}\sin^{2017}(x) = \left(\lim_{x\to a}\sin(x)\right)^{2017}$$

REMARK 1. Note that *all these properties also hold for the one-sided limits.*

REMARK 2. The analogues of the *laws 1-3 also hold when f and g are vector functions* (the product in Law 3 should be interpreted as a dot product).

$$\vec{r}_1(t) = \langle t, \cos\pi t + 2\rangle, \quad \vec{r}_2(t) = \langle 1-t, t\sin(\tfrac{\pi}{2}t)\rangle$$

$$\lim_{t\to 1}\vec{r}_1(t) = \langle 1, \cos\pi + 2\rangle = \langle 1, 1\rangle$$

$$\lim_{t\to 1}\vec{r}_2(t) = \langle 1-1, 1\cdot\sin\tfrac{\pi}{2}\rangle = \langle 0, 1\rangle$$

exists

$$\lim_{t\to 1}\vec{r}_1(t)\cdot\vec{r}_2(t) = \lim_{t\to 1}\vec{r}_1(t)\cdot\lim_{t\to 1}\vec{r}_2(t) =$$

$$= \langle 1, 1\rangle\cdot\langle 0, 1\rangle = 1$$

EXAMPLE 3. *Compute the limit:*

$$\lim_{x\to -1}(7x^3 - 5) \overset{1}{=} \lim_{x\to -1}(7x^3) - \lim_{x\to -1} 5 \overset{2}{\underset{5}{=}} 7\lim_{x\to -1} x^3 - 5 \overset{8}{=} \underbrace{7\cdot(-1)^3 - 5}_{f(-1)} = \boxed{-12}$$

Note: $f(x) = 7x^3 - 5$

Then $\lim_{x\to -1} f(x) = f(-1)$

Note $x = -1$ belongs to the domain of $f(x)$.

REMARK 4. The function from Example 3 also satisfies "direct substitution property":

$$\lim_{x\to a} f(x) = f(a).$$

Later we will say that such functions are *continuous*. Note that in both examples it was important that a in the domain of f.

EXAMPLE 5. *Compute the limit:*

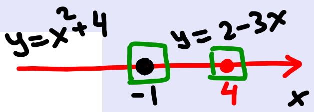
$$\lim_{x\to 1}\frac{x-1}{x^2-4x+3}$$

factor

$$= \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x-3)}$$

$$= \lim_{x\to 1}\frac{1}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$$

EXAMPLE 6. *Given*



$$g(x) = \begin{cases} x^2+4, & \text{if } x \le -1 \\ 2-3x & \text{if } x > -1 \end{cases}$$

Compute the limits:

(a) $\lim_{x\to 4} g(x) = \lim_{x\to 4}(2-3x) = 2-3\cdot 4 = -10$

(b) $\lim_{x\to -1} g(x)$

Consider one sided limits :

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1}(x^2+4) = (-1)^2+4 = 5$$

$(x<-1)$

$$\|$$

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+}(2-3x) = 2-3(-1) = 5$$

$(x>-1)$

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^+} g(x) = 5 \Rightarrow \boxed{\lim_{x\to -1} g(x) = 5}$$

$-(a-b) = b-a$

EXAMPLE 7. *Evaluate these limits.*

(a) $\lim_{x\to 4} \dfrac{x^{-1}-0.25}{x-4} = \lim_{x\to 4} \dfrac{\frac{1}{x}-\frac{1}{4}}{x-4} = \lim_{x\to 4} \dfrac{\frac{4-x}{4x}}{x-4}$

$= \lim_{x\to 4} \dfrac{4-x}{4x(x-4)} = \dfrac{1}{4}\lim_{x\to 4} \dfrac{-\cancel{(x-4)}}{x\cancel{(x-4)}} = -\dfrac{1}{4}\lim_{x\to 4}\dfrac{1}{x}$

$$\boxed{\dfrac{\frac{a}{b}}{c} = \dfrac{a}{bc}}$$

$= -\dfrac{1}{4}\cdot\dfrac{1}{4} = -\dfrac{1}{16}$

$(a+b)^2 = a^2+2ab+b^2$

(b) $\lim_{x\to 0} \dfrac{(x+5)^2-25}{x} = \lim_{x\to 0} \dfrac{x^2+10x+\cancel{25}-\cancel{25}}{x} = \lim_{x\to 0}\dfrac{\cancel{x}(x+10)}{\cancel{x}} = 0+10 = 10$

$\Vert$ Way 2 $\quad \underset{x+5}{a}^2 - \underset{5}{b}^2 = (a+b)(a-b)$

$\lim_{x\to 0} \dfrac{(x+5+5)\cancel{(x+5-5)}}{\cancel{x}} = \lim_{x\to 0}(x+10) = 10$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

(c) $\lim\limits_{x\to 0^-} \left\{ \frac{1}{x} - \frac{1}{|x|} \right\} = \lim\limits_{x\to 0^-} \left\{ \frac{1}{x} - \left(-\frac{1}{x}\right) \right\} = \lim\limits_{x\to 0^-} \frac{1}{2x} = -\infty$

$x<0 \Rightarrow |x| = -x \Rightarrow \frac{1}{|x|} = -\frac{1}{x}$

Remark The limit in (c) DNE and our answer indicates that the graph $y = \frac{1}{x} - \frac{1}{|x|}$ has a vert. asymptote $x=0$.

(d) $\lim\limits_{x\to -1} \frac{|x+1|}{x+1} = \lim\limits_{u\to 0} \frac{|u|}{u}$ DNE, because

$\boxed{\begin{array}{l} \text{Denote } u = x+1 \\ \text{Then } u \to 0 \text{ if } x \to -1 \end{array}}$

$\lim\limits_{\substack{u\to 0^- \\ u<0}} \frac{|u|}{u} = \lim\limits_{u\to 0^-} \frac{-u}{u} = \lim\limits_{u\to 0^-} (-1) = -1$

$\neq$

$\lim\limits_{\substack{u\to 0^+ \\ u>0}} \frac{|u|}{u} = \lim\limits_{u\to 0^+} \frac{u}{u} = 1$

Remark

$$\boxed{\text{sign}(u) = \frac{|u|}{u} = \begin{cases} 1, & u>0 \\ -1, & u<0 \end{cases}}$$

$$\frac{\sqrt{a}-\sqrt{b}}{1} = \frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = \frac{(\sqrt{a})^2-(\sqrt{b})^2}{\sqrt{a}+\sqrt{b}}$$

$$= \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

$$\frac{(\sqrt{a}-c)(\sqrt{a}+c)}{\sqrt{a}+c}$$

(e) $\displaystyle \lim_{x\to 0} \frac{\sqrt{6-x}-\sqrt{6}}{x}$

Multiply by conjugate

$$\lim_{x\to 0} \frac{(\sqrt{6-x}-\sqrt{6})(\sqrt{6-x}+\sqrt{6})}{x(\sqrt{6-x}+\sqrt{6})}$$

$$= \lim_{x\to 0} \frac{(\sqrt{6-x})^2-(\sqrt{6})^2}{x(\sqrt{6-x}+\sqrt{6})} = \lim_{x\to 0} \frac{6-x-6}{x(\sqrt{6-x}+\sqrt{6})}$$

$$= \lim_{x\to 0} \frac{-x}{x(\sqrt{6-x}+\sqrt{6})} = -\lim_{x\to 0} \frac{1}{\sqrt{6-x}+\sqrt{6}}$$

$$= -\frac{1}{\sqrt{6-0}+\sqrt{6}} = -\frac{1}{2\sqrt{6}} = -\frac{\sqrt{6}}{12}$$

Conclusion from the above examples:

To calculate the limit of $f(x)$ as $x \to a$:

*PLUG IN $x = a$ if a is in the domain of f. DIRECT SUB

*Otherwise "FACTOR" or "MULTIPLY BY CONJUGATE" and then plug in.

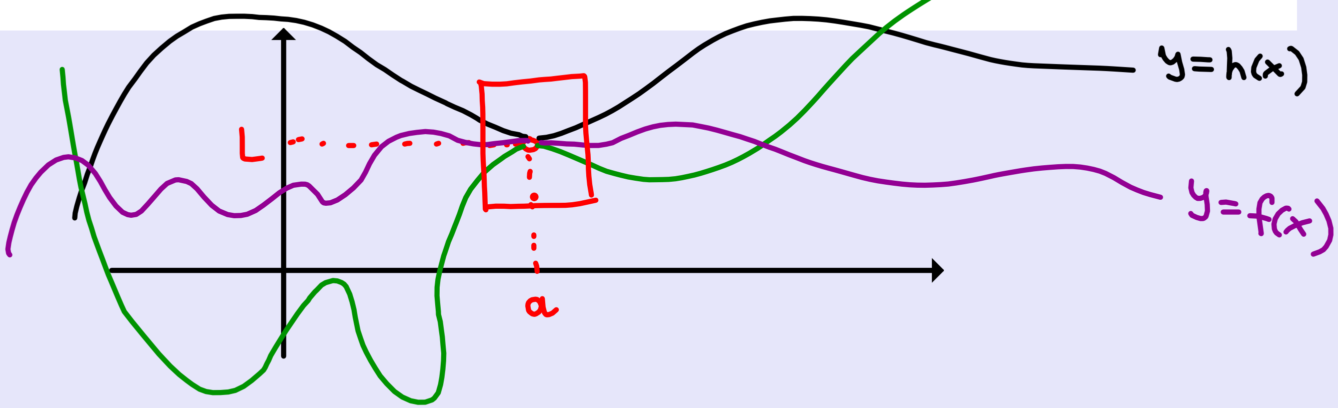
*Consider one sided limits if necessary.

Squeeze Theorem. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$g(x) \leq f(x) \leq h(x)$$

and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then

$$\lim_{x\to a} f(x) = L.$$



Corollary. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$|f(x)| \leq h(x) \quad (equivalently, \quad -h(x) \leq f(x) \leq h(x))$$

and $\boxed{\lim_{x\to a} h(x) = 0.}$ Then

$$\Downarrow$$

$$\lim_{x\to a} (-h(x)) = 0$$

$$\lim_{x\to a} f(x) = 0.$$

$$-h(x) \leq f(x) \leq h(x)$$

$$\begin{array}{ccccc} -h(x) & \leq & f(x) & \leq & h(x) \\ \downarrow & & \downarrow{\scriptstyle x\to a} & & \downarrow \\ 0 & & 0 & & 0 \end{array} \qquad x \to a$$

EXAMPLE 11. *Given* $\overbrace{3x}^{g(x)} \leq f(x) \leq \overbrace{x^3 + 2}^{h(x)}$ *for* $0 \leq x \leq 2$. *Find* $\lim_{x \to 1} f(x)$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (3x) = 3 \cdot 1 = 3$$

$$\lim_{x \to 1} h(x) = \lim_{x \to 1} (x^3 + 2) = 1^3 + 2 = 3$$

By Squeeze Theorem, we conclude that

$$\lim_{x \to 1} f(x) = 3$$

EXAMPLE 9. *Evaluate:*

(a) $\lim_{x \to 0} x \sin \frac{1}{x}$

$-1 \le \sin t \le 1$

or $|\sin t| \le 1$

If $x \neq 0$, $|\sin \frac{1}{x}| \le 1$

Then

$$\underbrace{\left| x \sin \frac{1}{x} \right|}_{f(x)} = |x| \cdot \left| \sin \frac{1}{x} \right| \le |x| \cdot 1 = \underbrace{|x|}_{h(x)}$$

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} |x| = 0$$

So, by Corollary of the Squeeze theorem,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

(b) $\lim_{t \to 0} (t^5) \cos^3(\frac{1}{t^2})$

If $t \neq 0$, then $\left| \cos^3 \frac{1}{t^2} \right| = \left| \cos \frac{1}{t^2} \right|^3 \le 1^3 = 1$

$$\left| t^5 \cos^3 \frac{1}{t^2} \right| = |t^5| \cdot \left| \cos^3 \frac{1}{t^2} \right| \le |t^5| \cdot 1 = |t^5|$$

Since $\lim_{t \to 0} t^5 = 0$, $\lim_{t \to 0} t^5 \cos^3 \frac{1}{t^2} = 0$

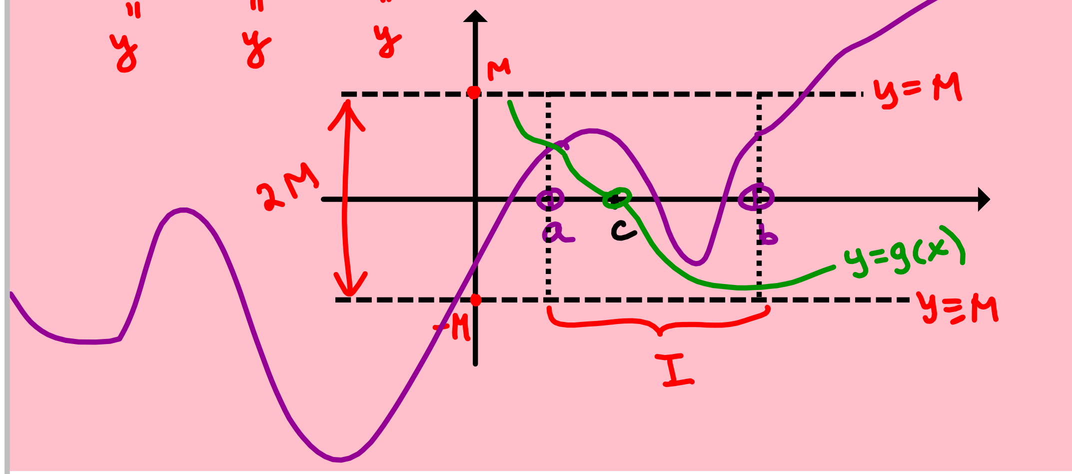
by Squeeze Theorem (corollary).

DEFINITION 10. *A function f is called* **bounded** *on an open interval I $=(a,b)$, if there exists a number M such that $|f(x)| \le M$ for all x in I.*

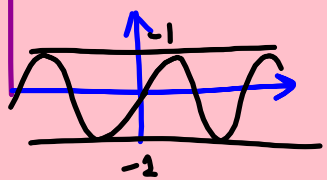
$$-M \le f(x) \le M \quad \text{for all } x \text{ in } (a,b)$$

$$\overset{\shortparallel}{y} \qquad \overset{\shortparallel}{y} \qquad \overset{\shortparallel}{y}$$

y=f(x)



F. ex. $f(x) = \sin x$ is an example of bdd everywhere function.



EXAMPLE 11. *Let f be a bounded function on an open interval I containing the point $x = c$ and g be a function defined on I, but not necessarily at $x = c$. Find $\lim_{x\to c}(f(x)g(x))$ if it is given that $\lim_{x\to c} g(x) = 0$.*

f is bdd on I $\Rightarrow$ there is a constant $M > 0$

such that $|f(x)| \le M$.

$$\big|f(x)\,g(x)\big| \le \underbrace{|f(x)|}_{\le M} \cdot |g(x)| \le M\,\underbrace{|g(x)|}_{\substack{\downarrow\, x\to c \\ 0}}$$

By Corollary of the Squeeze Theorem,

$$\lim_{x\to c} f(x)\,g(x) = 0.$$