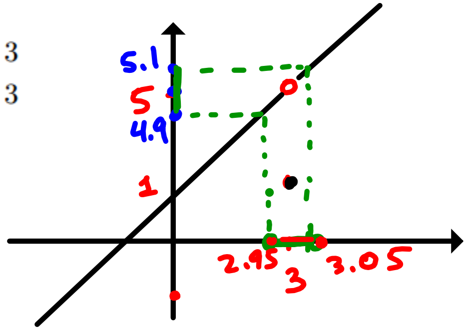
Section 2.4: The Precise Definition of a Limit ($\varepsilon - \delta$ definition)

Question: What does it mean $\lim_{x \to a} f(x) = L$? To motivate the precise definition of limit, consider the function

$$f(x) = \begin{cases} 2x - 1, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$$



- What is $\lim_{x \to 3} f(x)$?

  $\lim_{x \to 3} f(x) = 5$

- **Problem 1** *How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?* (δ)

  – The distance from x to 3 is $|x-3|$

  – The distance from $f(x)$ to 5 is $|f(x)-5|$

- Reformulation of problem 1: *Find a number δ such that*

  $|f(x)-5| < 0.1$, if $|x-3| < \delta$, $x \neq 3$

  If $\delta = \frac{0.1}{2} = 0.05$, then

Indeed,

$$|f(x)-5| = |(2x-1)-5| = |2x-6| = 2|x-3| < 2\delta = 2 \cdot 0.05 = 0.1$$

Thus an answer to the Problem 1 is given by $\delta =$ **0.05**; that is, if x is within a distance of **0.05** from 3, then $f(x)$ will be within a distance of **0.1** from 5.

- **Problem 2** *How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.01?*

Find a number δ such that if $|x-3|<\delta$, then $|f(x)-5|<0.01$

$\delta=\frac{0.01}{2}=0.005.$ $\quad |f(x)-5|=2|x-3|<2\delta=2\cdot 0.005=0.01$

- **Problem 3** *How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.001?*

$\delta=\frac{0.001}{2}=0.0005$

If $|x-3|<0.0005$, then

$|f(x)-5|=2|x-3|<2\cdot 0.0005=0.001$

- **Problem 4** *How close to 3 does x have to be so that $f(x)$ differs from 5 by less than an arbitrary positive number ε?*

$\varepsilon>0$

$$|f(x)-5|<\varepsilon \quad \text{if} \quad 0<|x-3|<\delta=\frac{\varepsilon}{2}. \tag{1}$$

In other words, we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x\neq 3$). This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3. Note that (1) can be rewritten as follows:

$|f(x)-5|<\varepsilon \quad$ if $\quad 0<|x-3|<\delta$

$|x-3|<\delta \quad (x\neq 3)$

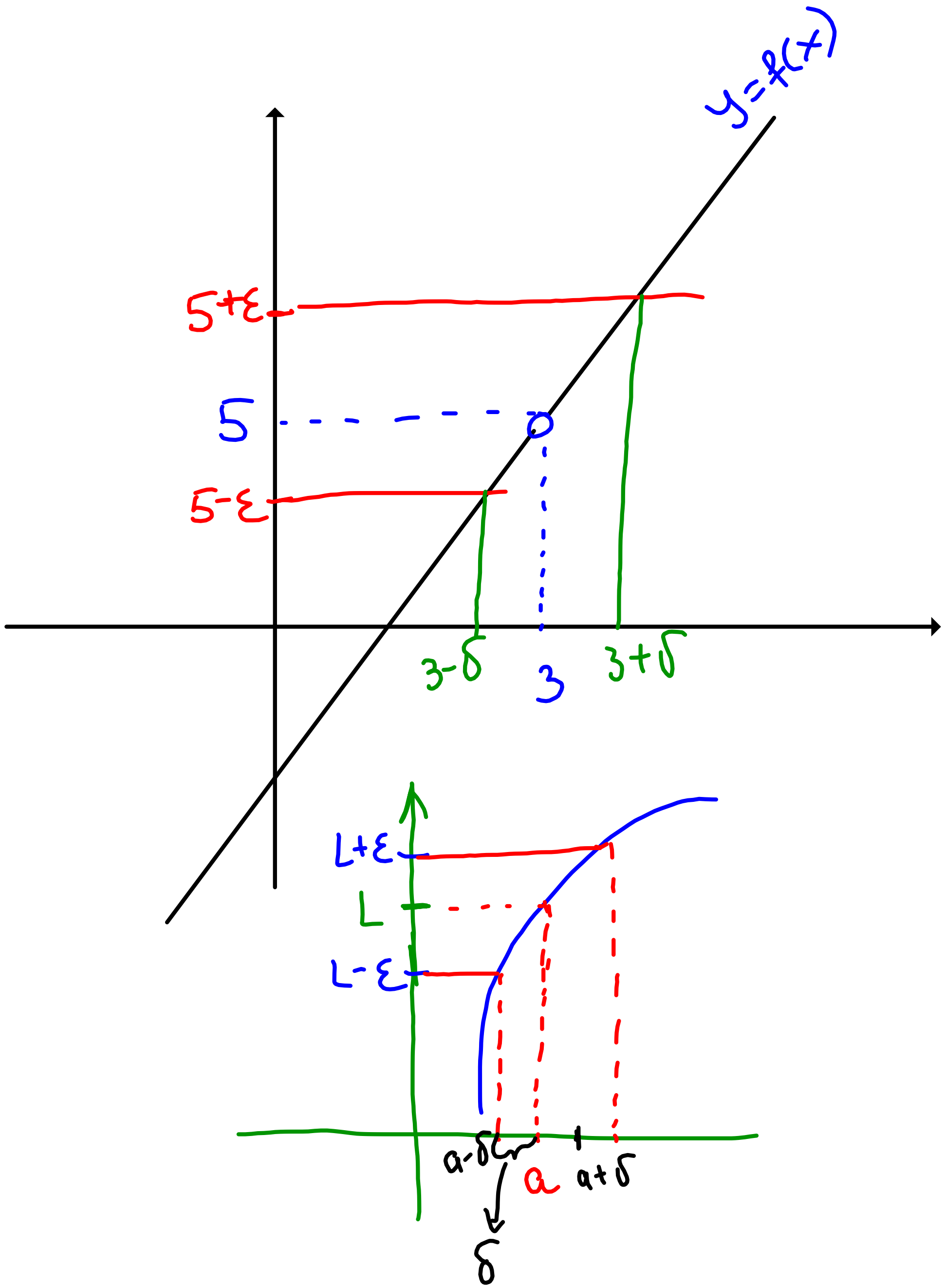
$-\varepsilon<f(x)-5<\varepsilon$

$-\delta<x-3<\delta$

$+5 \quad +5 \quad +5 \qquad +3 \quad +3 \quad +3$

$5-\varepsilon<f(x)<5+\varepsilon \quad$ if $\quad 3-\delta<x<3+\delta$

$y \quad y \quad y$


y=f(x)
5+ε
5
5-ε
3-δ
3
3+δ
L+ε
L
L-ε
a-δ
a
a+δ
δ

$0 < |x-a| < \delta$

DEFINITION 1. *Let $f(x)$ be a function defined for all x in some open interval containing the number a, except possibly at a itself. Then we say that* **the limit of $f(x)$ as x approaches a is L**, *and we write*

$$\lim_{x \to a} f(x) = L,$$

if for every number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

OR if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

OR if $a-\delta < x < a+\delta$ $(x \neq a)$, then $L-\varepsilon < f(x) < L+\varepsilon$

REMARK 2. For a limit from the right we need only assume that $f(x)$ is defined on an interval (a, b) extending to the right of a and that the ϵ condition is met for x in an interval $a < x < a + \delta$ extending to the right of a. A similar adjustment must be made for a limit from the left.

A general form of a limit proof

$\lim_{x \to a} f(x) = L$

Assume that we are given a positive number ϵ, and we try to prove that we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

There are two things to do: given $\varepsilon > 0$. Guess δ

1. Preliminary analysis of the problem (guessing a value for δ);
2. Proof (showing that the δ works). Show $|f(x) - L| < \varepsilon$, if $0<|x-a|<\delta$

Note that *the value of δ is not unique.* Namely, once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number $\delta_1, \delta_1 < \delta$, will also fulfill those requirements.

EXAMPLE 3. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 4} \overbrace{(3x-1)}^{f(x)} = \overbrace{11}^{L}$.

$= a$

① Guessing δ : if $0 < |x-4| < \delta$ then $\underbrace{|f(x)-11| < \varepsilon}$.

Given $\varepsilon > 0$.

$$|f(x)-11| = |(3x-1)-11| = |3x-12| =$$

$$= 3\underbrace{|x-4|} < \boxed{3\delta < \varepsilon}$$

$$\Downarrow$$

$$\delta < \frac{\varepsilon}{3}$$

Guess: $\boxed{\delta = \frac{\varepsilon}{3}}$

② Proof Given $\varepsilon > 0$, choose $\boxed{\delta = \frac{\varepsilon}{3}. \quad (**)}$

If $\boxed{0 < |x-4| < \delta}$ (*), then

$$|f(x)-11| = |(3x-1)-11| =$$

$$= |3x-12| = 3|x-4| \overset{(*)}{<} 3\delta \overset{(**)}{=} 3\cdot\frac{\varepsilon}{3} = \varepsilon.$$

So, if $0 < |x-4| < \delta$, then $|f(x)-11| < \varepsilon$, which means that

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(3x-1) = 11$$

by "ε-δ definition". □

EXAMPLE 4. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 5} x^2 = 25$.

① Guessing δ: if $0<|x-5|<\delta$, then $|x^2-25|<\varepsilon$

Given $\varepsilon>0$

$$|x^2-25| = |(x-5)(x+5)| = \underbrace{|x-5|}_{<\delta}\cdot|x+5| < \delta|x+5|$$

To estimate $|x+5|$ assume that $\delta<1$. Then

$|x-5|<\delta<1 \Rightarrow |x-5|<1 \Rightarrow$

$-1<x-5<1 \Rightarrow 4<x\le 6 \Rightarrow$

+5 +5 +5 +5 +5 +5

$9<x+5<11 \Rightarrow |x+5|<11$

$$|x^2-25|<\delta|x+5|<11\delta<\varepsilon$$

$$\delta<\frac{\varepsilon}{11}<1$$

So, our guess is

$$\delta=\min\left\{\frac{\varepsilon}{11},1\right\}$$

② **Proof** Given $\varepsilon > 0$, choose $\delta = \min\{\frac{\varepsilon}{11}, 1\}$.

If $\boxed{0 < |x-5| < \delta}^{(*)}$, then

$$|f(x)-25| = |x^2-25| = |x-5| \cdot |x+5| \overset{(*)}{<} \delta|x+5| \qquad (**)$$

Since $|x-5| < \delta \le 1$, we get

$$|x-5| < 1 \Rightarrow \underset{+5}{-1} < \underset{+5}{x-5} < \underset{+5}{1} \Rightarrow$$

$$\Rightarrow -1+5 < x-5+5 < 1+5 \Rightarrow$$

$$\Rightarrow \underset{+5}{4} < \underset{+5}{x} < \underset{+5}{6} \Rightarrow 9 < x+5 < 11 \Rightarrow$$

$$\Rightarrow |x+5| < 11 \qquad (***)$$

Continuing $(**)$, we get

$$|x^2-25| < \delta|x+5| \overset{(***)}{<} 11\delta <$$

$$< 11 \cdot \frac{\varepsilon}{11} = \varepsilon.$$

So, if $0 < |x-5| < \delta$, then $|x^2-25| < \varepsilon$, which means that $\lim_{x \to 5} x^2 = 25$ by "ε-δ definition". $\square$