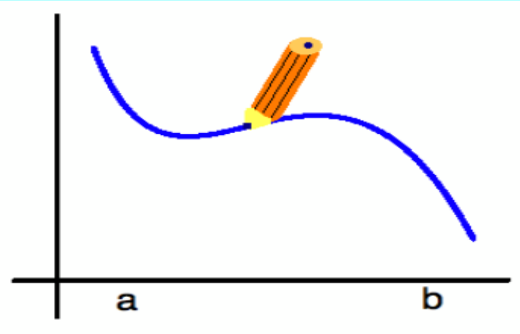
Section 2.5:Continuity

DEFINITION 1. *A function* $f(x)$ *is* **continuous** *at* $x = a$ *if* $\boxed{\lim_{x\to a} f(x) = f(a)}$ *More implicitly: if* f *is continuous at* a *then*

1. $f(a)$ *is defined (i.e.* a *is in the domain of* f*);*
2. $\lim_{x\to a} f(x)$ *exists.*
3. $\lim_{x\to a} f(x) = f(a)$.

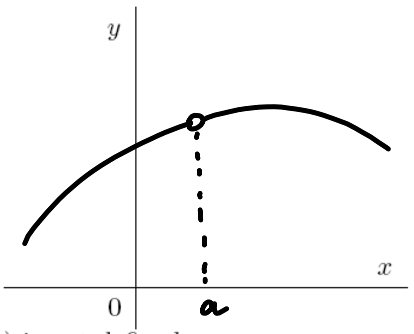
A function is said to be continuous on the interval $[a, b]$ *if it is continuous at each point in the interval.*

Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

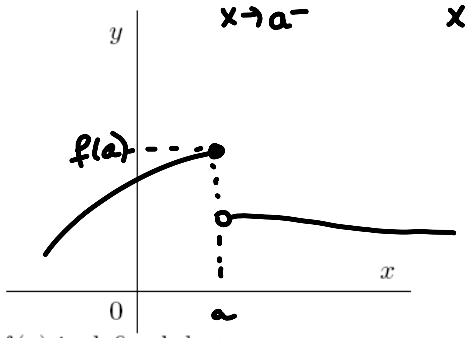


REASONS FOR BEING DISCONTINUOUS:

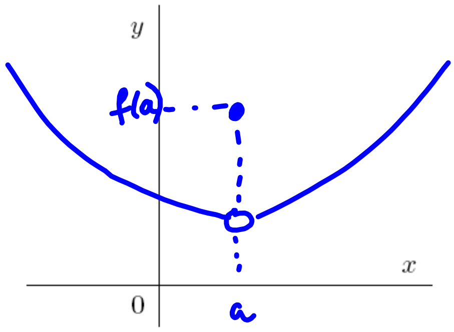
$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$



$f(a)$ is not defined
(i.e. a is not in the domain of f)



$f(a)$ is defined, but
the limit as $x \to a$ DNE

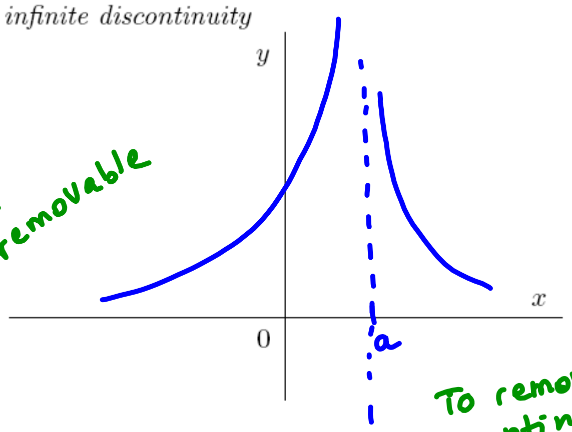


$f(a)$ is defined and $\lim_{x \to a} f(x)$ exists,
but $\lim_{x \to a} f(x) \neq f(a)$

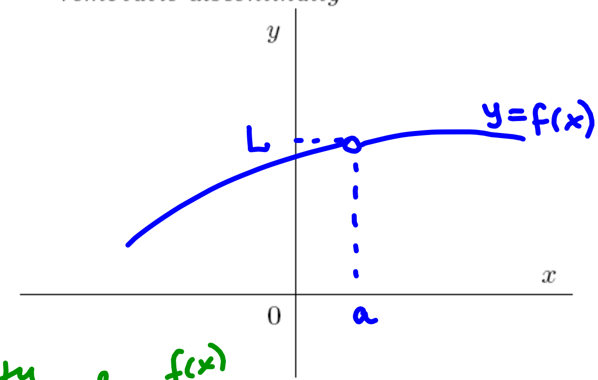
Classification of discontinuities:

infinite discontinuity

not removable

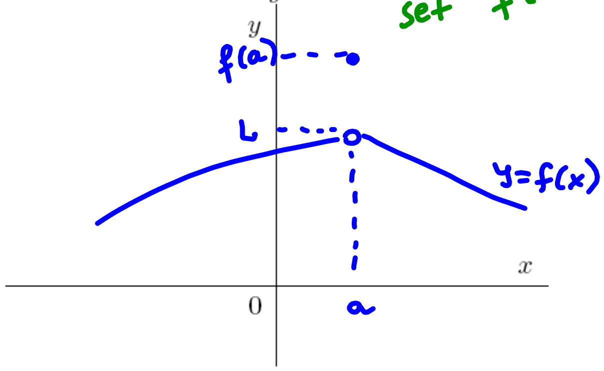


removable discontinuity

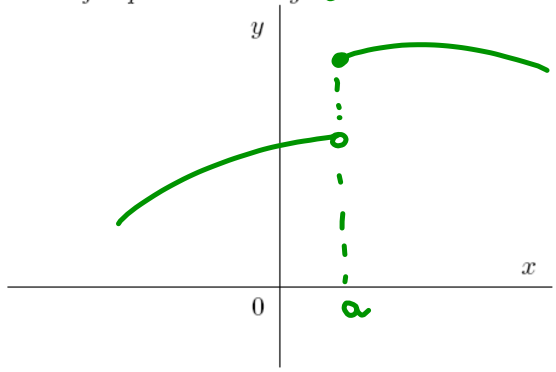


To remove discontinuity set $f(a) = L = \lim_{x \to a} f(x)$

removable discontinuity



jump discontinuity (not removable)



EXAMPLE 2. *Explain why each function is discontinuous at the given point:*

(a) $f(x) = \dfrac{2x}{x-3}, \quad x = 3$

$f(3)$ is undefined ($x=3$ is not in the domain of $f(x)$)

(b) $f(x) = \begin{cases} \dfrac{x^2-2x+1}{x-1} & \text{if } x \neq 1 \\ 5 & \text{if } x = 1, \end{cases} \quad x = 1$

$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)^2}{x-1} = \lim\limits_{x\to 1}(x-1) = 0 \neq f(1) = 5$

DEFINITION 3. *A function f is* **continuous from the right** *at $x = a$ if*

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left** *at a if*

$$\lim_{x \to a^-} f(x) = f(a).$$

REMARK 4. *Functions* ***continuous on an interval*** *if it is continuous at every number in the interval.* At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.

EXAMPLE 5. *Find the interval(s) where $f(x) = \sqrt{9 - x^2}$ is continuous.*

First find the domain of f:

$$9 - x^2 \geqslant 0$$

$$9 \geqslant x^2$$

$$\sqrt{x^2} \le \sqrt{9}$$

$$|x| \le 3$$

domain $\boxed{-3 \le x \le 3}$

$x = a$ and $-3 < x < 3$

$$\lim_{x \to a} f(x) = \sqrt{9 - a^2} = f(a)$$

$$\lim_{x \to 3^-} f(x) = \sqrt{9 - 3^2} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \sqrt{9 - (-3)^2} = 0 = f(-3)$$

Conclusion: $f(x)$ is continuous on $\boxed{[-3, 3]}$

EXAMPLE 6. *Find the constant c that makes g continuous on $(-\infty, \infty)$:*

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx^2 - 1 & \text{if } x \geq 4 \end{cases}$$

Since $x^2 - c^2$ and $cx^2 - 1$ are polynomials (i.e. continuous functions), $g(x)$ is continuous at every point, except, maybe, $x = 4$.

To garantee that $g(x)$ is continuous at $x=4$, we need that

$$\underbrace{\lim_{x \to 4^-}}_{x<4} g(x) = \underbrace{\lim_{x \to 4^+}}_{x>4} g(x) \ (= g(4))$$

$$\lim_{x \to 4^-} (x^2 - c^2) = \lim_{x \to 4^+} (cx^2 - 1)$$

$$4^2 - c^2 = c \cdot 4^2 - 1$$

$$16 - c^2 = 16c - 1$$

$$c^2 + 16c - 17 = 0$$

$$(c-1)(c+17) = 0$$

$$\boxed{c = 1 \quad \text{OR} \quad c = -17}$$

EXAMPLE 7. *For each of the following,* (1) *find all discontinuities,* (2) *classify them by using limits,* (3) *give the continuity interval(s) for the corresponding function.* (4) *If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(a) $f(x) = \dfrac{x^2 - 9}{x^4 - 81}$

(1) $x^4 = 81 \Rightarrow x = \pm 3$ discontinuities

(2) $$\lim_{x \to \pm 3} \frac{x^2 - 9}{x^4 - 81} = \lim_{x \to \pm 3} \frac{\cancel{x^2 - 9}}{\cancel{(x^2 - 9)}(x^2 + 9)}$$

$$= \frac{1}{(\pm 3)^2 + 9} = \frac{1}{18} \text{ exists}$$

But $f(\pm 3)$ is undefined

$f(x)$ has removable discontinuities at $x = \pm 3$.

(3) -3 3

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

(4) $$g(x) = \begin{cases} f(x), & x \neq \pm 3 \\ \frac{1}{18}, & x = \pm 3 \end{cases}$$

(b) $f(x) = \dfrac{7}{x+12}$

① $x+12=0 \Rightarrow x=-12$ discontinuity.

② $\displaystyle\lim_{x \to 12^-} \frac{7}{x+12} = \infty$

$x < 12$

at $x=-12$ $f(x)$ has infinite discontinuity.

③ $(-\infty, -12) \cup (-12, \infty)$

EXAMPLE 7. *For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(c) $f(x) = \begin{cases} x^2 + x & \text{if } x < 2 \\ 8 - x & \text{if } x > 2 \\ 4 & \text{if } x = 2 \end{cases}$

Note that $f(x)$ is defined for all x.

Investigate $x=2$ by using limits:

$$\lim_{x \to 2^-} f(x) \underset{x<2}{=} \lim_{x\to 2^-} (x^2+x) = 2^2+2 = 6$$

$$\lim_{x \to 2^+} f(x) \underset{x>2}{=} \lim_{x\to 2^+} (8-x) = 8-2 = 6$$

$$\lim_{x\to 2} f(x) = 6 \neq 4 = f(2)$$

Thus $f(x)$ has a $\boxed{\text{removable discontinuity}}$ at $\boxed{x=2}$.

If $x \neq 2$ then $f(x)$ is a polynomial, i.e. $f(x)$ is continuous there.

Interval of continuity: $(-\infty, 2) \cup (2, \infty)$

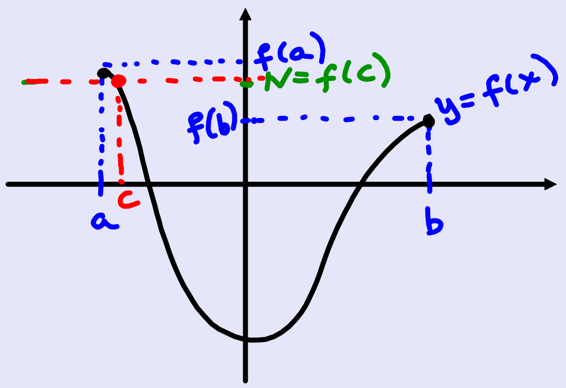
To remove the discontinuity at $x = 2$, define

$$g(x) = \begin{cases} f(x), & x \neq 2 \\ 6, & x = 2 \end{cases} = \begin{cases} x^2+x, & x \leq 2 \\ 8-x, & x > 2 \end{cases}$$

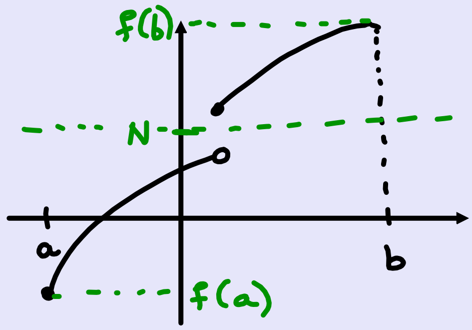
($g(x)$ is continuous everywhere).

Intermediate Value Theorem: If $f(x)$ is continuous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $\underbrace{a < c < b}_{(a,b)}$, so that $f(c) = N$.

$f(a) < N < f(b)$ OR $f(b) < N < f(a)$



continuity is important



The IVT implies existence of a solution $x=c$ for the equation $f(x) = N$.

And if $f(x)$ is a polynomial, then $x=c$ is a root of the polynomial $f(x) - N$.

IVT

Intermediate Value Theorem: If $f(x)$ is continous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a < c < b$, so that $f(c) = N$.

EXAMPLE 8. *If $f(x) = x^5 - 2x^3 + x^2 + 2$, show ~~there~~ that is a number c so that $f(c) = 1$.* $= N$

Since the function $f(x)$ is continuous every where (as a polynomial), in order to apply the IVT, it is sufficient to find a and b such that the number 1 is strictly between $f(a)$ and $f(b)$.

x	$f(x)$
0	$2 > 1$
-1	$-1+2+1+2 > 1$
-2	$-32+16+4+2 < 0 < 1$

We obtained that

$f(0) > 1$ and $f(-2) < 1$

Thus, there exists a number c such that

$-2 < c < 0$ and $f(c) = 1$.

or $\boxed{-1 < c < 0}$

Intermediate Value Theorem: If $f(x)$ is continous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a < c < b$, so that $f(c) = N$.

EXAMPLE 9. *Show that following equation has a solution (a root) between 1 and 2:*

$$3x^3 - 2x^2 - 2x - 5 = 0.$$

x	$f(x)$
1	$3-2-2-5<0$
2	$3\cdot8-2\cdot4-2\cdot2-5>0$

Let $f(x) = 3x^3 - 2x^2 - 2x - 5$.
Since $f(x)$ is continuous on $[1,2]$ (as a polynomial), we will apply the IVT. By IVT it is sufficient to find a and b such that $1 < a, b < 2$ and $f(a) \cdot f(b) < 0$ (i.e. $f(a)$ and $f(b)$ have opposite signs).

If $a = 1$ and $b = 2$ then $f(a) = f(1) < 0$ and $f(b) = f(2) > 0$.

So, $f(x) = 0$ have a solution on $(1, 2)$.