Section 3.2: Differentiation formulas

The properties and formulas in this section will be given in both "prime" notation and "fraction" notation.

PROPERTIES:

1. **Constant rule**: If f is a constant function, $f(x) = c$, then $f'(x) = 0$, or $\frac{\mathrm{d}c}{\mathrm{d}x} = 0$.

$$f'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

$(2017)' = 0$

2. **Power rule:** If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$, or $\dfrac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.

$(x^{2017})' = 2017\, x^{2017-1} = 2017\, x^{2016}$

$(x^{-2017})' = -2017\, x^{-2017-1} = -2017\, x^{-2018}$

$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}\, x^{\frac{1}{2}-1} = \frac{1}{2}\, x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\dfrac{\mathrm{d}\left(\frac{1}{x}\right)}{\mathrm{d}x} = \dfrac{d(x^{-1})}{dx} = (-1)x^{-1-1} = -x^{-2} = \frac{1}{x^2}$

$(x)' = (x^1)' = 1 \cdot x^{1-1} = x^0 = 1$

$$\boxed{\begin{aligned} x' &= 1 \\ (\sqrt{x})' &= \frac{1}{2\sqrt{x}} \\ \left(\frac{1}{x}\right)' &= -\frac{1}{x^2} \end{aligned}}$$

3. **Constant multiple rule:** If c is a constant and $f'(x)$ exists then

$$(cf(x))' = cf'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}.$$

$(7\sqrt{x})' = 7(\sqrt{x})' = \frac{7}{2\sqrt{x}}$

4. **Sum/Difference rule**: If $f'(x)$ and $g'(x)$ exists then

$$(f(x) \pm g(x))' = f'(x) \pm g'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(f \pm g) = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x}.$$

$$\frac{\mathrm{d}(3e^x - 5\cos x)}{\mathrm{d}x} = (3e^x - 5\cos x)' = (3e^x)' - (5\cos x)' = 3(e^x)' - 5(\cos x)'$$

COROLLARY 1. *If c and d are real constants, then*

$$(cf(x) + dg(x))' = c\, f'(x) + d\, g'(x)$$

EXAMPLE 2. *Find the derivatives of the following functions:*

(a) $f'(x) = (x^{10} + 3x^5 - 12x + 2017 - \pi^5)'$

$= (x^{10})' + 3(x^5)' - 12(x)' + (2017 - \pi^5)'$

$= 10x^9 + 3 \cdot 5x^4 - 12 \cdot 1 + 0 \quad = 10x^9 + 15x^4 - 12$

(b) $g(t) = (1 + \sqrt{t})^2 = 1^2 + (\sqrt{t})^2 + 2\sqrt{t} = 1 + t + 2\sqrt{t}$

$g'(t) = 1' + t' + 2(\sqrt{t})' = 0 + 1 + 2 \cdot \frac{1}{2\sqrt{t}} = 1 + \frac{1}{\sqrt{t}}$

$(fg)' = f'g + fg'$

5. **Product rule:** If $f'(x)$ and $g'(x)$ exists then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}.$$

Way I: $(x^1 \cdot x^{\frac{1}{2}})' = (x^{\frac{3}{2}})' = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

$(\underbrace{x}_{F}\underbrace{\sqrt{x}}_{G})' =$

Way II: $F'G + FG' = x'\sqrt{x} + x(\sqrt{x})' = \sqrt{x} + x\frac{1}{2\sqrt{x}} = \sqrt{x} + \frac{\sqrt{x}}{2} = \frac{3}{2}\sqrt{x}$

EXAMPLE 3. *Find the derivatives of* $f(x) = \underbrace{(x^4 - 3x^2 + 11)}_{F}\underbrace{(3x^3 - 5x^2 + 22)}_{G}$

$f'(x) = (x^4 - 3x^2 + 11)'(3x^3 - 5x^2 + 22) + (x^4 - 3x^2 + 11)(3x^3 - 5x^2 + 22)'$

$= (4x^3 - 6x)(3x^3 - 5x^2 + 22) + (x^4 - 3x^2 + 11)(9x^2 - 10x)$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

6. **Quotient rule:** If $f'(x)$ and $g'(x)$ exists then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{[g(x)]^2}.$$

EXAMPLE 4. *Find the derivatives of* $g(z) = \dfrac{4 - z^2 \} F}{4 + z^2 \} G}$

$$\left(\frac{F}{G}\right)' = \frac{F'G - FG'}{G^2}$$

$$g'(z) = \frac{(4-z^2)'(4+z^2) - (4-z^2)(4+z^2)'}{(4+z^2)^2}$$

$$= \frac{-2z(4+z^2) - (4-z^2)\cdot 2z}{(4+z^2)^2}$$

$$= \frac{-8z - \cancel{2z^3} - 8z + \cancel{2z^3}}{(4+z^2)^2} = -\frac{16z}{(4+z^2)^2}$$

EXAMPLE 5. *Given* $f(x) = x^3 - 5x^2 + 6x - 3$

(a) *Find the equation of the tangent line to the graph of* $f(x)$ *at the point* $(1, -1)$.

$(1 = x_0,\ -1 = y_0 = f(x_0))$

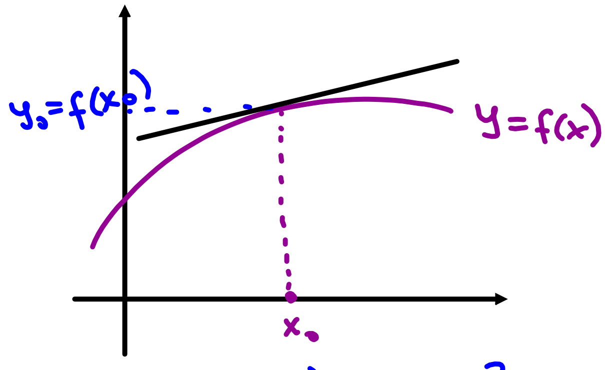
$f'(x) = 3x^2 - 10x + 6$

$m = f'(1) = 3 - 10 + 6 = -1$

$y - (-1) = -1(x - 1)$

$y + 1 = -x + 1$

$\boxed{y = -x}$



$m = f'(x_0)$

$\left.\begin{array}{l} m = f'(x_0) \\ y - y_0 = m(x - x_0) \end{array}\right\} \Rightarrow$

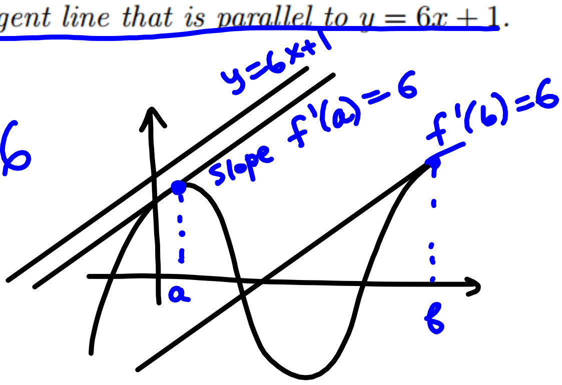
$\boxed{y - f(x_0) = f'(x_0)(x - x_0)}$

equation of tangent to the graph of $y = f(x)$ at x_0.

(b) *Find the value(s) of* x *where* $f(x)$ *has a tangent line that is parallel to* $y = 6x + 1$.

Find x such that

slope at x $= f'(x) = \left(\text{slope of } y = 6x+1\right) = 6$



$f'(x) = \underbrace{3x^2 - 10x + 6 = 6}$

$3x^2 - 10x + \cancel{6} = \cancel{6}$

$3x^2 - 10x = 0$

$x(3x - 10) = 0$

$x = 0$ OR $3x - 10 = 0$

$x = \frac{10}{3}$

Answer: $0;\ \frac{10}{3}$

EXAMPLE 6. *The functions f and g satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	5	12
3	1	2	-2	8

Find the indicated quantity: $(FG)' = F'G + FG'$

(a) $h'(3)$ *if* $h(x) = (3x^2+1)g(x)$

$$h'(x) = \Big(\underbrace{(3x^2+1)}_{F}\,\underbrace{g(x)}_{G}\Big)' =$$

$$= (3x^2+1)'\,g(x) + (3x^2+1)g'(x)$$

$$= 6x\,g(x) + (3x^2+1)\,g'(x)$$

$$h'(3) = 6\cdot 3\cdot g(3) + (3\cdot 3^2+1)\,g'(3)$$

$$= 18\cdot(-2) + 28\cdot 8$$

$$= 4(-9 + 7\cdot 8) = 4\cdot 47$$

$$= 160 + 28 = \boxed{188}$$

(b) $H'(1)$ *if* $H(x) = \dfrac{x^2}{f(x)} = \dfrac{F}{G}$

$$\left(\frac{F}{G}\right)' = \frac{F'G - FG'}{G^2}$$

$$H'(x) = \frac{(x^2)'\,f(x) - x^2\cdot f'(x)}{[f(x)]^2}$$

$$H'(x) = \frac{2x\,f(x) - x^2 f'(x)}{[f(x)]^2}$$

$$H'(1) = \frac{2\cdot 1\cdot f(1) - 1^2 f'(1)}{[f(1)]^2}$$

$$H'(1) = \frac{2(-5) - 8}{(-5)^2} = -\frac{18}{25}$$

EXAMPLE 7. *Let* $f(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$

(a) *Give a formula for* f'.

Check continuity at -1 and 1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-1-2x) = 1 = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to -1} f(x) = 1 = f(-1)$$

So, $f(x)$ is continuous at $x = -1$.

$$\lim_{x \to 1^-} f(x) = 1^2 = 1 = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1} f(x) = 1 = f(1)$$

So, $f(x)$ is continuous at $x = 1$

To find derivatives at $x = \pm 1$:

$$\left.\begin{aligned} f'_-(-1) &= (-1-2x)' = -2 \\ f'_+(-1) &= (x^2)'\Big|_{x=-1} = 2x\Big|_{x=-1} = -2 \end{aligned}\right\} \Rightarrow$$

$$\boxed{f'(-1) = -2}$$

$$f'_-(1) = (x^2)'\Big|_{x=1} = 2x\Big|_{x=1} = 2$$

$$f'_+(1) = x'\Big|_{x=1} = 1$$

$\neq$

$f'(1)$ DNE

$$f'(x) = \begin{cases} -2 & , \text{ if } x < -1 \\ 2x & , \text{ if } -1 \le x < 1 \\ 1 & , \text{ if } x > 1 \end{cases}$$

(b) *For what value(s) of x the function is not differentiable?*

$x = 1$

EXAMPLE 8. *Let*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 2 \\ ax+b & \text{if} \quad x > 2 \end{cases}$$

Find the values of a and b that makes f differentiable everywhere.

First note that f is differentiable when $x>2$ and $x<2$ (as a polynomial).

If $x=2$ then f must be continuous there.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \quad \text{and exist}$$

$$\lim_{x\to 2^-} x^2 = \lim_{x\to 2^+} (ax+b)$$

$$2^2 = a\cdot 2 + b \Rightarrow \boxed{b = 4-2a}$$

For differentiability we also need

$$f'_-(2) = f'_+(2)$$

$$(x^2)'\big|_{x=2} = (ax+b)\big|_{x=2}$$

$$2x\big|_{x=2} = a\big|_{x=2}$$

$$\boxed{a=4} \Rightarrow b = 4-2a = 4-2\cdot 4 = -4.$$

Answer: $a=4$, $b=-4$.

EXAMPLE 9. *A ball is thrown into the air. Its position at time t is given by*

$$\vec{r}(t) = \langle 2t, 10t - t^2 \rangle .$$

(a) *Find the velocity of the ball at time $t = 2$.*

$\vec{v}(t) = \vec{r}\,'(t) = \langle (2t)', (10t - t^2)' \rangle = \langle 2, 10 - 2t \rangle$

$\vec{v}(2) = \langle 2, 6 \rangle$

(b) *Find the speed of the ball at time $t = 2$.*

speed at $t=2$ $= |\vec{v}(2)| = |\langle 2, 6 \rangle| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$