Section 3.5: Chain Rule

Question: How to find the derivatives of the following functions:

$$y = (x^6 + 4x^2 + 12)^{15}; \qquad y = \sec(12x^2) + \tan^3(x) \qquad y = \sqrt[3]{4+x}$$

Review of Composite Functions:

$$[f \circ g](x) = f(g(x))$$

If $f(x) = x^{15}$ and $g(x) = x^6 + 4x^2 + 12$ then

$$[f \circ g](x) = f(g(x)) = f(x^6 + 4x^2 + 12) = (x^6 + 4x^2 + 12)^{15}$$

Conversely, if $[f \circ g](x) = \sec(12x^2)$ then $f(x) = \sec x$ and $g(x) = 12x^2$

The CHAIN RULE: *If the derivatives $g'(x)$ and $f'(x)$ both exist, and $F = f \circ g$ is the composite defined by*

$$F(x) = f(\underbrace{g(x)}_{u})$$

$F' = f'u'$

then

$$F'(x) = f'(g(x))g'(x).$$

In Leibniz notation: If the derivatives of $y = f(u)$ and $u = g(x)$ both exist then

$$y = f(g(x))$$

is differentiable function of x and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

$y = f(g(x)) = f(u)$

$y = f(x)$	$u(x)$	$f(u)$	$\frac{dy}{dx} = f'(u) \cdot u'(x)$
$y = (x^6 + 4x^2 + 12)^{15}$	$u = x^6 + 4x^2 + 12$ $u' = 6x^5 + 8x$	$y = u^{15}$ $y' = 15u^{14} =$ $= 15(x^6 + 4x^2 + 12)^{14}$	$\frac{dy}{dx} = 15(x^6 + 4x^2 + 12)^{14}(6x^5 + 8x)$
$y = \sec(12x^2)$	$u = 12x^2$ $u' = 24x$	$y = \sec u$ $y' = \sec u \tan u$	$\frac{dy}{dx} =$ $= 24x \sec(12x^2)\tan(12x^2)$
$y = \tan^3(x)$ $= (\tan x)^3$	$u = \tan x$ $u' = \sec^2 x$	$y = u^3$ $y' = 3u^2$	$\frac{dy}{dx} = 3\tan^2 x \sec^2 x$
$y = \sqrt[3]{4+x}$	$u = 4 + x$ $u' = 1$	$y = \sqrt[3]{u} = u^{\frac{1}{3}}$ $y' = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-\frac{2}{3}}$ $= \frac{1}{3\sqrt[3]{u^2}}$	$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{u^2}} \cdot 1$ $= \frac{1}{3\sqrt[3]{(4+x)^2}}$
$y = [g(x)]^n$	$u = g(x)$ $u' = g'(x)$	$y = u^n$ $y' = nu^{n-1} =$ $n[g(x)]^{n-1}$	$\frac{dy}{dx} = n[g(x)]^{n-1} \cdot g'(x)$ Generalized Power Rule

EXAMPLE 1. *Find the derivative:*

(a) $f(x) = \dfrac{1}{(x^3+5x^2+12)^{2017}} = (\underbrace{x^3+5x^2+12}_{u})^{-2017}$

$f'(x) = -2017\,(x^3+5x^2+12)^{-2017-1} \cdot (x^3+5x^2+12)'$

$= -2017\,(x^3+5x^2+12)^{-2018}(3x^2+10x)$

(b) $h(x) = x^8(3\sqrt{x}-11)^8 = \left(x(3\sqrt{x}-11)\right)^8 = (3x\sqrt{x}-11x)^8$

$= (\underbrace{3x^{\frac{3}{2}}-11x}_{u})^8$

$h'(x) = 8\,(3x^{\frac{3}{2}}-11x)^{8-1}\left(3\cdot\frac{3}{2}x^{\frac{3}{2}-1} - 11\right)$

$= 8\,(3x\sqrt{x}-11x)^7\left(\frac{9}{2}\sqrt{x}-11\right)$

(c) $f(x) = \cos(5x) + \cos^5 x = \cos(\underbrace{5x}_{u_1}) + (\underbrace{\cos x}_{u_2})^5$

$f'(x) = -\sin(5x)(5x)' + 5\cos^4 x\,(\cos x)'$

$= -5\sin(5x) - 5\cos^4 x \sin x$

EXAMPLE 2. *Find F' and G' if*

$$F(x) = f(\underbrace{\sin x}_{u}), \qquad G(x) = \sin(\underbrace{f(x)}_{u}),$$

where $f(x)$ is a differentiable function.

$$F'(x) = f'(\sin x) \cdot (\sin x)' = f'(\sin x)\cos x$$

$$\boxed{\frac{d}{dx}\Big(f(\sin x)\Big) = f'(\sin x)\cos x}$$

$$G'(x) = \cos(f(x)) \cdot f'(x)$$

$$\boxed{\frac{d}{dx}\big(\sin(f(x))\big) = \cos(f(x))f'(x)}$$

EXAMPLE 3. *Let $f(x)$ and $g(x)$ be given differentiable functions satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	3	12
3	1	2	-2	8

Suppose that $h = f \circ g$. Find $h'(1)$.

$h(x) = (f \circ g)(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(1) = f'(g(1)) \cdot g'(1)$

$h'(1) = f'(3) \cdot 12 = 2 \cdot 12 = 24$