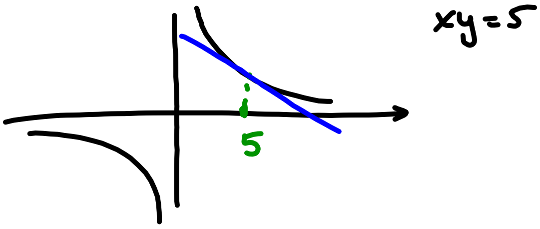


3.6: Implicit differentiation

EXAMPLE 1. *Find y' if the $y = y(x)$ satisfies the equation $xy = 5$ for all values of x in its domain and evaluate $y'(5)$.*

Solution 1 (by explicit differentiation):

$xy = 5 \Rightarrow y(x) = \frac{5}{x} \Rightarrow y'(x) = \left(\frac{5}{x}\right)' = -\frac{5}{x^2}$

$y'(5) = -\frac{5}{5^2} = \boxed{-\frac{1}{5}}$

Solution 2 (by implicit differentiation):

$xy(x) = 5$

$(fg)' = f'g + fg'$

$\frac{d}{dx}\left(\underbrace{x}_{f} \cdot \underbrace{y(x)}_{g}\right) = \frac{d}{dx}(5)$

$x'\,y(x) + x\,y'(x) = 0$

$y(x) + x\,y'(x) \Rightarrow y'(x) = -\frac{y(x)}{x}$

Find $y(5)$:

$5 \cdot y(5) = 5$

$y(5) = 1.$

$y'(5) = -\frac{y(5)}{5} = \boxed{-\frac{1}{5}}$

EXAMPLE 2. **(a)** *If* $x^2 + y^2 = 16$ *find* $\dfrac{dy}{dx}$.

$y = y(x)$

$$x^2 + [y(x)]^2 = 16$$

Differentiate both sides w.r.t. x.

$$\frac{d}{dx}\left[x^2 + [\underbrace{y(x)}_{u}]^2\right] = \frac{d}{dx}[16]$$

$$2x + 2\,y(x)\,y'(x) = 0$$

$$x + y\,y' = 0$$

$$y\,y' = -x \Rightarrow y' = -\frac{x}{y}$$

$x_0 \quad y_0$

(b) *Find the equation of the tangent line to* $x^2 + y^2 = 16$ *at the point* $(2, 2\sqrt{3})$.

$$y - y_0 = m(x - x_0)$$

where $m = y'(x_0)$

In our case:

$x_0 = 2$, $y_0 = 2\sqrt{3}$

$$m = y'(2) \overset{(a)}{=} -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\boxed{y - 2\sqrt{3} = -\frac{\sqrt{3}}{3}(x - 2)}$$

EXAMPLE 3. *Find* $\frac{dy}{dx}$ *if*

$$x^3 - \cot(xy^2) = x\cos y$$

$y = y(x)$

$$\frac{d}{dx}\left(x^3 - \cot\left(\underbrace{x\,(y(x))^2}_{u}\right)\right) = \frac{d}{dx}\left(\underbrace{x}_{f}\cdot\underbrace{\cos(y(x))}_{g}\right)$$ Product Rule

$$3x^2 - \left(-\csc^2(xy)\right)\cdot\frac{d}{dx}\left(\underbrace{x}_{f}\underbrace{(y(x))^2}_{g}\right) = x'\cos y + x\frac{d}{dx}(\cos(y(x)))$$

$$3x^2 + \csc^2(xy)\left(x'y^2 + x\frac{d}{dx}(y(x)^2)\right) = \cos y + x(-\sin y)y'$$

$$3x^2 + \csc^2(xy)\left(y^2 + x\cdot 2yy'\right) = \cos y - xy'\sin y$$

$$3x^2 + y^2\csc^2(xy) + 2xyy'\csc^2(xy) = \cos y - xy'\sin y$$

$$2xyy'\csc^2(xy) + xy'\sin y = \cos y - 3x^2 - y^2\csc^2(xy)$$

$$y'\left(2xy\csc^2(xy) + x\sin y\right) = \cos y - 3x^2 - y^2\csc^2(xy)$$

$$\frac{dy}{dx} = y' = \frac{\cos y - 3x^2 - y^2\csc^2(xy)}{2xy\csc^2(xy) + x\sin y}$$

EXAMPLE 4. *Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find* $\frac{dx}{dy}$, *if*

$$(x^2 + y^2)^5 = x^2 y^3$$

$x = x(y)$

$$\frac{d}{dy}\left(\left[\underbrace{[x(y)]^2 + y^2}_{u}\right]^5\right) = \frac{d}{dy}\left(\underbrace{[x(y)]^2}_{f} \cdot \underbrace{y^3}_{g}\right) \text{ product Rule}$$

$$5(x^2+y^2)^4 \frac{d}{dy}\left[[x(y)]^2 + y^2\right] = \frac{d}{dx}\left([x(y)]^2\right) y^3 + x^2 \cdot 3y^2$$

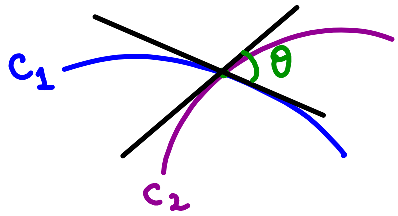
$$5(x^2+y^2)^4 (2x\,x' + 2y) = 2x\,x'\,y^3 + 3x^2y^2$$

$$10(x^2+y^2)^4 x\,x' + 10y(x^2+y^2)^4 = 2x\,x'\,y^3 + 3x^2y^2$$

$$10(x^2+y^2)^4 x\,x' - 2x\,x'\,y^3 = 3x^2y^2 - 10y(x^2+y^2)^4$$

$$x'\left(10x(x^2+y^2)^4 - 2xy^3\right) = 3x^2y^2 - 10y(x^2+y^2)^4$$

$$\frac{dx}{dy} = x' = \frac{3x^2y^2 - 10y(x^2+y^2)^4}{10x(x^2+y^2)^4 - 2xy^3}$$



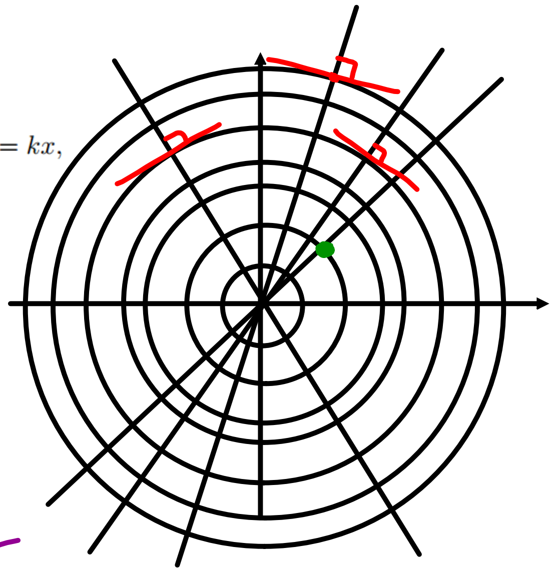
$C_1 \perp C_2$ if and only if $\theta = \frac{\pi}{2}$

DEFINITION 5. *Two curves are said to be* **orthogonal** *if at the point(s) of their intersection, their tangent lines are orthogonal(perpendicular). In this case we also say that the angle between these curves is* $\frac{\pi}{2}$.

`Illustration:` Consider two families of curves:

$$x^2 + y^2 = r^2, \quad y = kx,$$

where r and k are real parameters.



The angle between the curves from the above families at their points of intersection is the angle between corresponding tangent lines at these points.

Find their slopes:

$$x^2 + y^2 = r^2$$

$$\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d}{dx}(r^2)$$

$$2x + 2yy' = 0$$

$$x + yy' = 0$$

$$m_1 = y'(x) = -\frac{x}{y}$$

$$y = kx$$

$$m_2 = k = \frac{y}{x}$$

So, $m_1 \cdot m_2 = -\frac{x}{y} \cdot \frac{y}{x} = -1$

Thus these families of curves are orthogonal at their point of intersection.