3.8: Higher Derivatives

The derivative of a differentiable function f is also a function and it may have a derivative of its own:

$$(f')' = f'' \qquad \textbf{second derivative}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f'(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$$

`Alternative Notation:` If $y = f(x)$ then

$$y'' = f''(x) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \underbrace{D^2f(x)}. \quad = D(Df(x))$$

Similarly, the **third derivative** $f''' = (f'')'$ or

$$y^{(3)} = y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = D^3f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is denoted by $f^{(n)}(x)$:

$$y^{(n)} = f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right) = D^nf(x). = D(D^2f(x))$$

$$D^nf(x) = D^{n-k}(D^kf(x))$$

EXAMPLE 1. *If* $f(x) = x^5 + 3x + 1$ *find* $f^{(n)}(x)$

$$f'(x) = 5x^4 + 3$$

$$f''(x) = (f'(x))' = 20x^3$$

$$f'''(x) = ((f'(x))')' = 60x^2$$

$$f^{(4)}(x) = f^{IV}(x) = (((f'(x))')')' = (60x^2)' = 120x$$

$$f^{(5)}(x) = f^{V}(x) = ((((f'(x))')')')' = 120$$

$$f^{(6)}(x) = f^{VI}(x) = 0$$

all derivatives of order 6 and more are equal to 0.

$$f^{(k)}(x) = 0 \text{ for all } k \geq 6$$

CONCLUSION: *If* $p(x)$ *is a polynomial of degree* n *then,* $p^{(k)}(x) = 0$ *for* $k \geq n + 1$.

EXAMPLE 2. *Find* $D^{2013}\sin x$.

$f(x) = \sin x$
find $f^{(2013)}(x)$
or $\frac{d^{2013} f(x)}{dx^{2013}}$

cycle

$f(x) = \sin x$

$D^1 f(x) = (\sin x)' = \cos x$

$D^2 f(x) = D(\cos x) = -\sin x$

$D^3 f(x) = D(-\sin x) = -\cos x$

$D^4 f(x) = D(-\cos x) = -(-\sin x) = \sin x$

$\sin x = f(x) = D^4 f(x) = D^8 f(x) = D^{12} f(x) = \ldots = D^{4k} f(x)$

where $k \geq 1$

$2013 = 503 \cdot 4 + 1$

$D^{2012} f(x) = D^{503 \cdot 4} f(x) = f(x) = \sin x$

$D^{2013} f(x) = D(D^{2012} f(x)) = D(\sin x) = \cos x.$

Acceleration: If $s(t)$ is the position of an object then the acceleration of the object is the first derivative of the velocity (consequently, the acceleration is the second derivative of the position function.) $v(t) = s'(t)$

$$a(t) = v'(t) = s''(t).$$

EXAMPLE 3. *If $s(t) = t^3 - \frac{9}{2}t^2 - 30t + 12$ is the position of a moving object at time t (where $s(t)$ is measured in feet and t is measured in seconds) find the acceleration at the times when the velocity is zero.*

Find t such that

$v(t) = s'(t) = \underbrace{3t^2 - 9t - 30 = 0}$

$t^2 - 3t - 10 = 0$

$(t-5)(t+2) = 0$

$t = 5$ OR $t = -2 < 0$ (disregard)

Find $a(5)$:

$a(t) = v'(t) = (3t^2 - 9t - 30)' = 6t - 9$

$a(5) = 6 \cdot 5 - 9 = 21 \ ft/s^2$

Implicit second derivatives:

EXAMPLE 4. *Find $y''(x)$ if $x^6 + y^6 = 66$.*

$$x^6 + y^6 = 66$$

$$\frac{d}{dx}\left(x^6 + (y(x))^6\right) = \frac{d}{dx}(66)$$

$$\cancel{6}x^5 + \cancel{6}y^5 y' = 0 \implies y' = -\frac{x^5}{y^5}$$

$$\frac{d}{dx}\left(x^5 + \underbrace{[y(x)]^5}_{f} \cdot \underbrace{y'(x)}_{g}\right) = \frac{d}{dx}(0)$$

$$\underbrace{5x^4 + \underbrace{5y^4 y'}_{f'} \cdot \underbrace{y'}_{g}} + \underbrace{y^5}_{f}\underbrace{y''}_{g'} = 0$$

$$y^5 y'' = -5\left(x^4 + y^4 (y')^2\right)$$

$$y'' = -\frac{5}{y^5}\left(x^4 + y^4 (y')^2\right)$$

$$y'' = -\frac{5}{y^5}\left(x^4 + y^4 \cdot \left(-\frac{x^5}{y^5}\right)^2\right)$$

$$y'' = -\frac{5}{y^5}\left(x^4 + \frac{y^4 x^{10}}{y^{10}}\right)$$

$$y'' = -\frac{5x^4}{y^5}\left(1 + \frac{x^6}{y^6}\right)$$

$$y'' = -\frac{5x^4}{y^5} \cdot \frac{y^6 + x^6}{y^6} = 66$$

$$y'' = \frac{-5x^4 \cdot 66}{y^{11}}$$

$$y'' = \frac{-330x^4}{y^{11}}$$