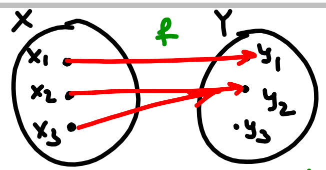
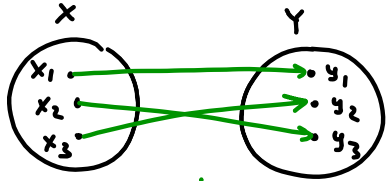


$x_2 \neq x_3$, but $f(x_2) = f(x_3) = y_2$
not one-to-one



one-two-one

4.2:Inverse Functions

injective

DEFINITION 1. *A function of domain X is said to be a* **one-to-one** *function if no two elements of X have the same image, i.e.*

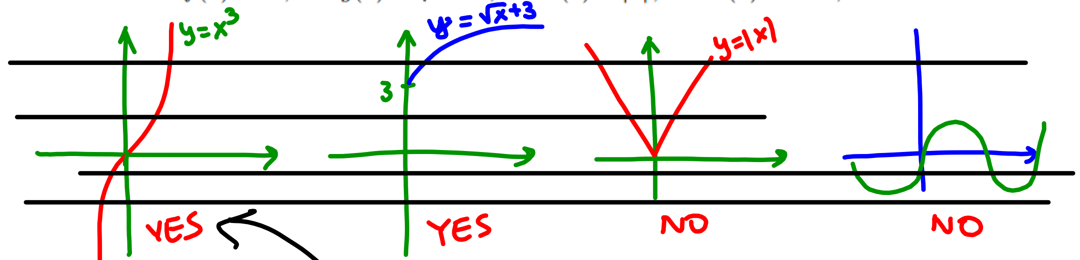
if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Equivalently, $\boxed{\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.}$

Horizontal line test:A function if one-to-one is and only if no horizontal line intersects its graph more once.

EXAMPLE 2. *Are the following functions one-to-one?*

$$f(x) = x^3, \qquad g(x) = \sqrt{x} + 3, \qquad u(x) = |x|, \qquad w(x) = \sin x,$$



YES YES NO NO

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$

$x_1 = x_2$

Consider $F(x) = x^2$

$F(x_1) = F(x_2)$

$x_1^2 = x_2^2$

$\sqrt{x_1^2} = \sqrt{x_2^2}$

$|x_1| = |x_2|$

$x_1 = x_2$ OR $x_1 = -x_2$

non-injective

$1 \neq -1$, but $F(1) = F(-1) = 1$.

EXAMPLE 3. *Prove that* $f(x) = \dfrac{x-3}{x+3}$ *is one-to-one* $(x \neq -3)$.

Proof

$$\boxed{f(x_1) = f(x_2)}$$

$$\frac{x_1-3}{x_1+3} = \frac{x_2-3}{x_2+3}$$

$$(x_1-3)(x_2+3) = (x_1+3)(x_2-3)$$

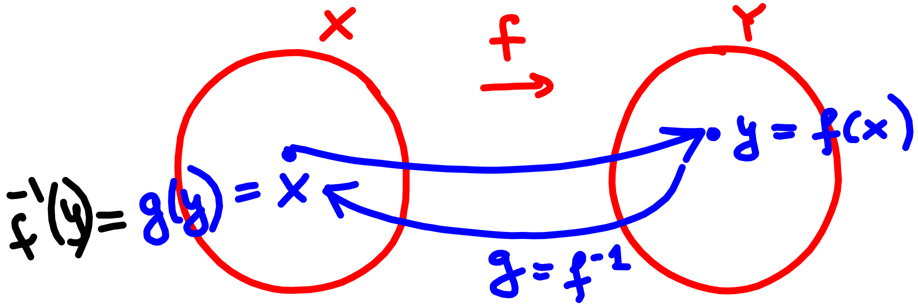
$$\cancel{x_1x_2} - 3x_2 + 3x_1 \cancel{-9} = \cancel{x_1x_2} + 3x_2 - 3x_1 \cancel{-9}$$

$$3x_1 + 3x_1 = 3x_2 + 3x_2$$

$$6x_1 = 6x_2$$

$$\boxed{x_1 = x_2}$$

So, f is one-to-one.



$$y = f(x) = f(g(y)) = (f \circ g)(y)$$

$$x = g(y) = g(f(x)) = (g \circ f)(x)$$

DEFINITION 4. *Let f be a one-to-one function with domain X and range Y. Then the inverse function f^{-1} has the domain Y and range X and is defined for any y in Y by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y.$$

REMARK 5. Reversing roles of x and y in the last formula we get:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

REMARK 6. If $y = f(x)$ is one-to-one function with the domain X and the range Y then

for every x in X $\quad f^{-1}(f(x)) = x$

and

for every x in Y $\quad f(f^{-1}(x)) = x$

Cancellation Laws

CAUTION: $f^{-1}(x)$ **does NOT mean** $\dfrac{1}{f(x)}$.

TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f*:*

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 7. *(cf. Example3) Find the inverse function of* $f(x) = \dfrac{x-3}{x+3}$. *Find the domain and the range of both* f *and* f^{-1}.

$$y = \frac{x-3}{x+3}$$

$$y(x+3) = x-3$$

$$xy + 3y = x - 3$$

$$3y + 3 = x - xy$$

$$3(y+1) = x(1-y)$$

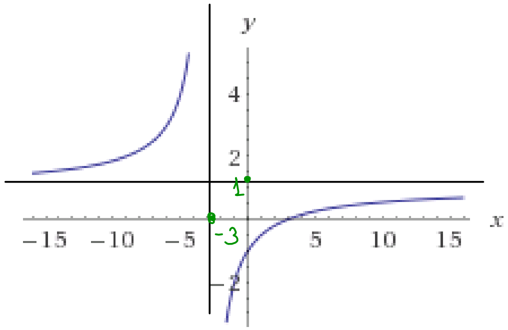
$$x = \frac{3(y+1)}{1-y}$$

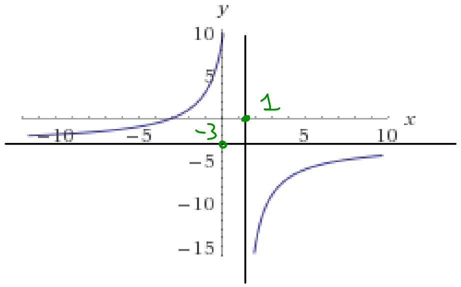
$$y = \boxed{\frac{3(x+1)}{1-x} = f^{-1}(x)}$$

	Domain	Range
Function f	$(-\infty,-3)\cup(-3,\infty)$	$(-\infty,1)\cup(1,\infty)$
Function f^{-1} (Inverse of f)	$(-\infty,1)\cup(1,\infty)$	$(-\infty,-3)\cup(-3,\infty)$

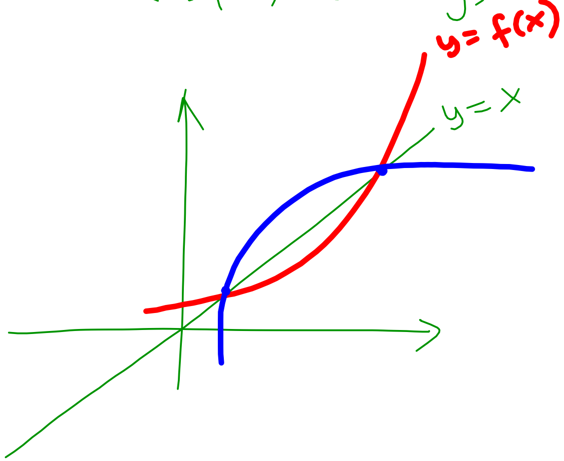
FACT: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

$$y = \frac{x-3}{x+3}$$





If (a, b) belongs to the graph $y = f(x)$,
then (b, a) belongs to the graph $y = f^{-1}(x)$.



THEOREM 8. *If f is a one-to-one differentiable function with inverse function $\underbrace{g = f^{-1}}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and*

$$\frac{d f^{-1}(a)}{dx} = g'(a) = \frac{1}{f'(g(a))}. = \frac{1}{f'(f^{-1}(a))}$$

Proof.

By Cancellation Law

$$f(f^{-1}(x)) = x \text{, or}$$

$$f(g(x)) = x$$

Differentiate both sides (use Chain Rule)

$$f'(g(x)) \cdot g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

If $x = a$ and $f'(g(a)) \neq 0$, then

$$g'(a) = \frac{1}{f'(g(a))}$$

EXAMPLE 9. *Suppose that g is inverse of f. Find $g'(a)$ where*

(a) $f(x) = \dfrac{x-3}{x+3}$, $a = 3$. *This problem was covered on recitation. Below is solution of a similar one.*

(a') $f(x) = \dfrac{2x-3}{x+3}$, $a = \dfrac{1}{2}$.

$g = f^{-1}$ Find $g'(\frac{1}{2})$

$g(\frac{1}{2}) = x$

$f^{-1}(\frac{1}{2}) = x$

Find x such that

$\frac{1}{2} = f(x)$

$\frac{1}{2} = \dfrac{2x-3}{x+3}$

$x+3 = 2(2x-3)$

$x+3 = 4x-6$

$3x = 9$

$\boxed{x=3} \Rightarrow \boxed{g(\frac{1}{2})=3}$

$$g'(\tfrac{1}{2}) = \frac{1}{f'(g(\frac{1}{2}))} = \frac{1}{f'(3)} = \frac{1}{\frac{1}{4}} = 4$$

Quotient Rule

$$f'(x) = \frac{d}{dx}\left(\frac{2x-3}{x+3}\right) = \frac{2(x+3) - (2x-3)}{(x+3)^2}$$

$$f'(3) = \frac{2\cdot 6 - 3}{6^2} = \frac{9}{36} = \frac{1}{4}$$

$$\boxed{g'(\tfrac{1}{2}) = 4}$$

(b) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$

$g = f^{-1}$ Find $g'(2)$.

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$\boxed{g'(2) = \frac{2}{3}}$

$g(2) = x$

OR $f^{-1}(2) = x \Rightarrow f(x) = 2$

Guess $x = 1 \Rightarrow f(1) = \sqrt{4} = 2$

So, $\boxed{g(2) = 1}$

Find $f'(1)$:

$$f'(x) = \frac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}} \Rightarrow f'(1) = \frac{3 + 2 + 1}{2\sqrt{4}} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

Find $g'(a)$ where $g = f^{-1}$

(c) $f(x) = 4 + 3x + e^{3(x-1)}$, $a = 8$.

$$g'(a) = \frac{1}{f'(g(a))}$$

$$g'(8) = \frac{1}{f'(g(8))}$$

$$g'(8) = \frac{1}{f'(1)}$$

$$\boxed{g'(8) = \frac{1}{6}}$$

Find $g(8)$:

$g(8) = x$

$f^{-1}(8) = x$

$8 = f(x)$

Find x s.t. $f(x) = 8$

Guess $\underset{x=1}{f(1) = 4 + 3 + e^0} = 8$

So, $\boxed{g(8) = 1}$

Find $f'(1)$:

$f'(x) = 0 + 3 + 3e^{3(x-1)}$

$f'(1) = 3 + 3 \cdot 1 = 6$