4.4: Derivatives of Logarithmic Functions

EXAMPLE 1. *Using Implicit Differentiation find the derivatives of the following function:*

(a) $f(x) = \ln x$

$y = \ln x$

$e^{y} = x$

$e^{y(x)} = x$

Differentiate both sides w.r.t. x.

$y' e^{y} = x'$

$y' = \frac{1}{e^{y}} = \frac{1}{x}$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

(b) $f(x) = a^x$

$y = a^x \xrightarrow{\text{way2}} \log_a y = \log_a a^x$

way

$\ln y = \ln a^x$

$\dfrac{\ln y}{\ln a} = x$

$\ln y = x \ln a$

$\ln y(x) = x \ln a$

Differentiate both sides w.r.t. x.

$\dfrac{y'}{y} = \ln a$

$y' = y \ln a = a^x \ln a$

$$\boxed{(a^x)' = a^x \ln a}$$

Combining the formulas obtained in Example 1 and Chain Rule one can get

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}a^{g(x)} = a^{g(x)} \ln a \; g'(x).$$

EXAMPLE 2. *Find the derivative:*

(a) $f(x) = \ln(\underbrace{\sin x}_{u=g(x)})$

$$f'(x) = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x} = \cot x$$

(b) $f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

$$f'(x) = \begin{cases} (\ln x)', & x > 0 \\ (\ln(-x))', & x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1), & x < 0 \end{cases} = \frac{1}{x}$$

$$(\ln|x|)' = \frac{1}{x}.$$

(c) $f(x) = 5^{\cot x}$

$$f'(x) = 5^{\cot x} \ln 5 \; (\cot x)'$$

$$f'(x) = 5^{\cot x} \ln 5 \, (-\csc^2 x)$$

$$f'(x) = -(\ln 5) 5^{\cot x} \csc^2 x$$

EXAMPLE 3. *Using the change of base formula, find the derivative formula for* $f(x) = \log_a x$ *and generalize it using the Chain Rule.*

$$f(x) = \log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \ln x$$

$$f'(x) = \frac{1}{\ln a} (\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

$$(\log_a g(x))' = \frac{g'(x)}{g(x) \ln a}$$

EXAMPLE 4. *Find the derivative of* $f(x) = \log_2(\overbrace{3 + x^2 + x^3}^{u=g(x)})$

$$f'(x) = \frac{(3+x^2+x^3)'}{(3+x^2+x^3)\ln 2} = \frac{2x+3x^2}{(3+x^2+x^3)\ln 2}$$

Logarithmic Differentiation can be used to find derivative of complicated functions involving products, quotients or powers.

STEPS IN LOGARITHMIC DIFFERENTIATION:

1. Take logarithms of both sides of an equation $y = f(x)$ and simplify (f.ex. split a product or quotient, etc.).
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.
4. Plug in $y = f(x)$.

EXAMPLE 5. *Find the derivative:* $y = (\cos x)^{\sin x}$

① $\ln y = \ln (\cos x)^{\sin x}$

$\ln y(x) = (\sin x)(\ln \cos x)$

② Differentiate both sides w.r.t. x.

$$\frac{y'}{y} = \cos x (\ln \cos x) + \sin x \frac{(\cos x)'}{\cos x}$$

$$\frac{y'}{y} = \cos x (\ln \cos x) - \frac{\sin^2 x}{\cos x}$$

③ $y' = y\left(\cos x (\ln(\cos x)) - \frac{\sin^2 x}{\cos x}\right)$

④ $y' = (\cos x)^{\sin x}\left(\cos x (\ln(\cos x)) - \frac{\sin^2 x}{\cos x}\right)$

Ex

$$y = \frac{(\tan x)\sqrt[3]{x^2+\cos x+5}\ \sqrt{2017-x-x\sqrt{x}}}{[\cos(\tan x^3)]^3}$$

$$\ln y = \ln(\tan x) + \frac{1}{3}\ln(x^2+\cos x+5) + \frac{1}{2}\ln(2017-x-x\sqrt{x})$$
$$- 3\ln(\cos(\tan x^3))$$

$\vdots$

$y' = \ldots$