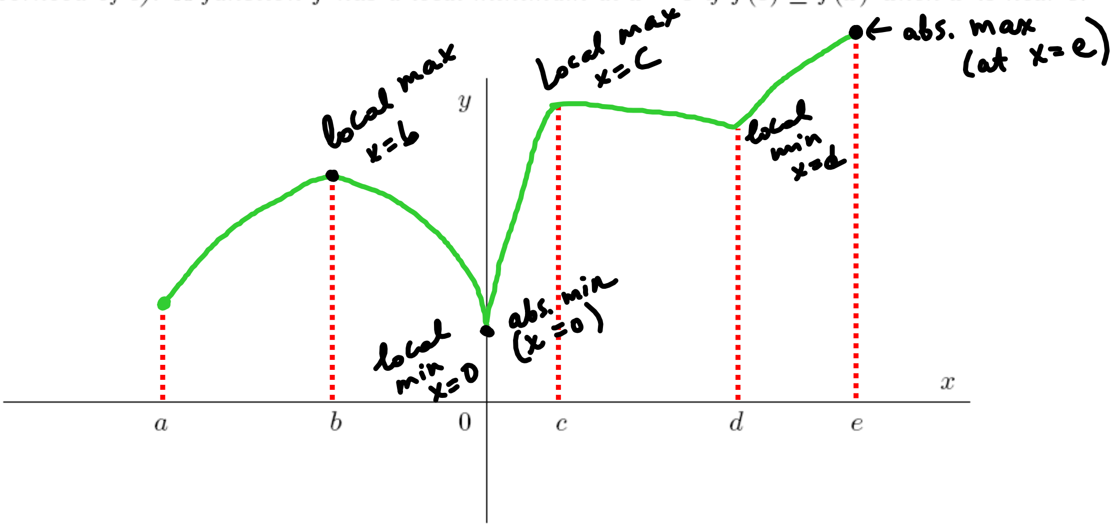
5.2: Maximum and Minimum Values

DEFINITION 1. *Let D be the domain of a function f.* an interval

- *A function f has an* **absolute maximum** *(or* **global maximum***) at $x = c$ if $f(c) \geq f(x)$ for all x in D. In this case, we call $f(c)$ the* **maximum value.**
- *A function f has an* **absolute minimum** *(or* **global minimum***) at $x = c$ if $f(c) \leq f(x)$ for all x in D. In this case, we call $f(c)$ the* **minimum value.**

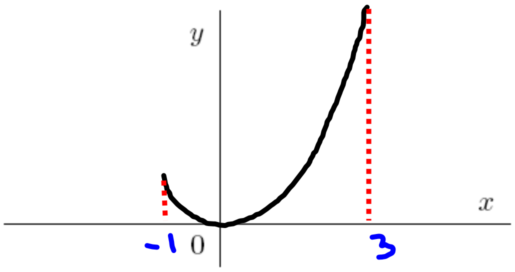
The maximum and minimum values of f on D are called the **extreme values** *of f.*

DEFINITION 2. *A function f has a local maximum at $x = c$ if $f(c) \geq f(x)$ when x is near c (i.e. in a neighborhood of c). A function f has a local minimum at $x = c$ if $f(c) \leq f(x)$ when x is near c.*



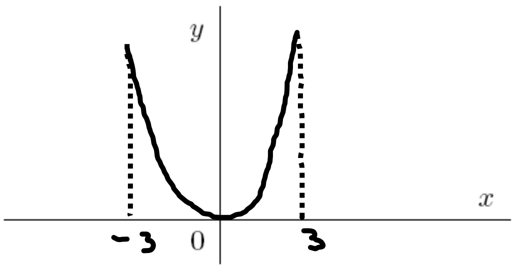
EXAMPLE 3. *Find the absolute and local extrema of f by sketching its graph:*

(a) $f(x) = x^2$, $-1 \le x \le 3$



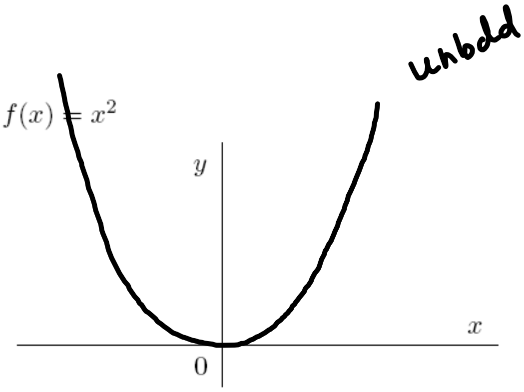
	Local	*Absolute*	*Value*
Maximum	no	x=3	f(3)=9
Minimum	x=0	x=0	f(0)=0

(b) $f(x) = x^2$, $-3 \le x \le 3$



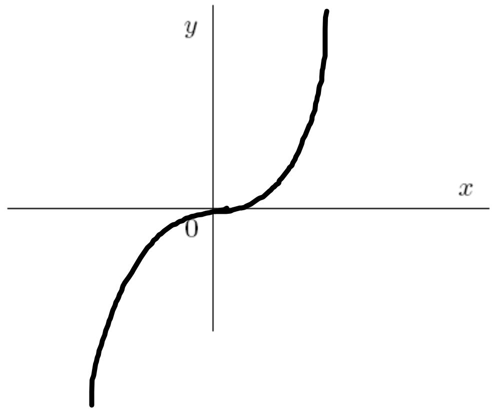
	Local	*Absolute*	*Value*
Maximum	no	x=3 or x=-3	f(±3)=9
Minimum	x=0	x=0	f(0)=0

(c) $f(x) = x^2$



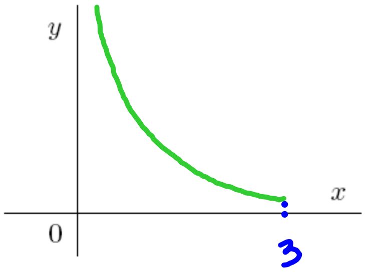
	Local	*Absolute*	*Value*
Maximum	no	no	no
Minimum	x=0	x=0	f(0)=0

(d) $f(x) = x^3$



	Local	*Absolute*	*Value*
Maximum	NO		
Minimum			

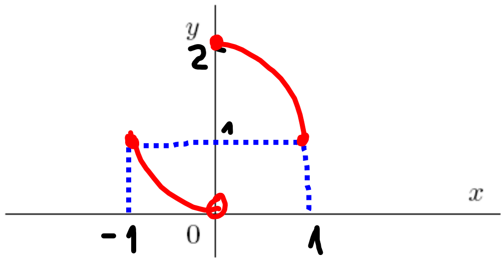
(e) $f(x) = \frac{1}{x}$, $0 < x \le 3$



	Local	*Absolute*	*Value*
Maximum	NO	NO	—
Minimum		$x=3$	$f(3)=\frac{1}{3}$

(f) $f(x) = \begin{cases} x^4 & \text{if} \quad -1 \le x < 0 \\ 2 - x^4 & \text{if} \quad 0 \le x \le 1 \end{cases}$

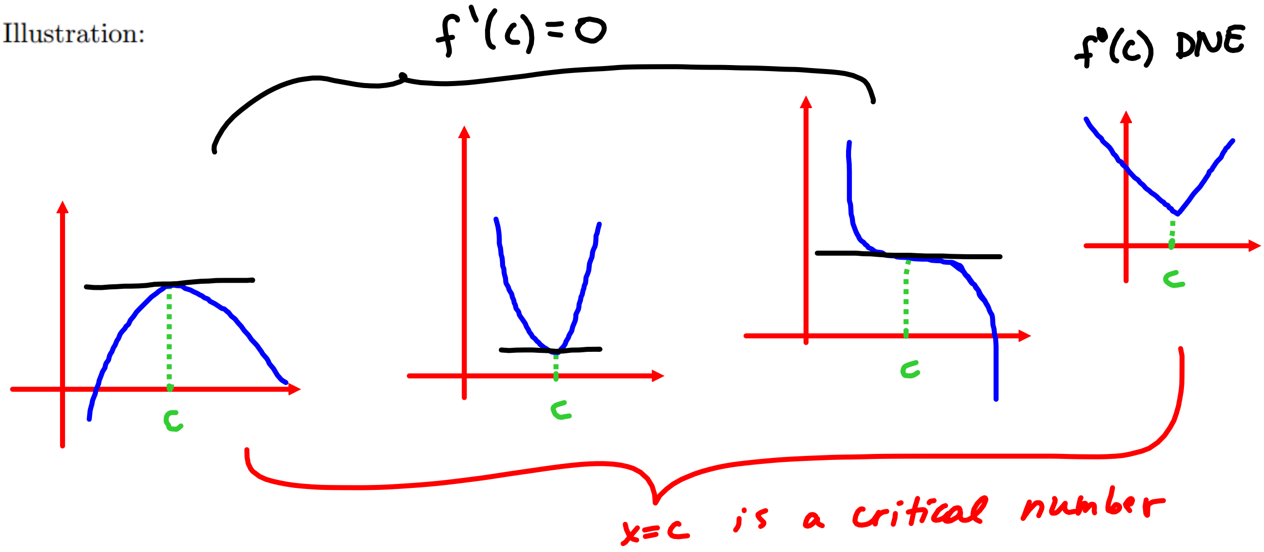
	Local	*Absolute*	*Value*
Maximum	NO	$x=0$	$f(0)=2$
Minimum		no	no



point

DEFINITION 4. *A* **critical number** *of $f(x)$ is a number c is in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

Illustration:



EXAMPLE 5. *Find the critical numbers of $f(x)$:*

(a) $f(x) = x \ln x$

$f'(x) = (x \ln x)' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$f'(x) = 0 \iff \ln x + 1 = 0$

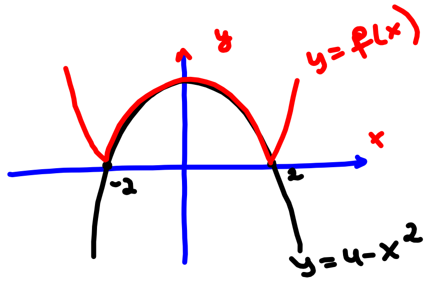
$\ln x = -1$

$x = e^{-1} = \frac{1}{e}$ in the domain of f.

$\boxed{x = \frac{1}{e}}$ is a critical number of f.

(b) $f(x) = |4 - x^2|$

First note that $f'(\pm 2)$ DNE



$$f(x) = \begin{cases} 4 - x^2, & -2 \le x \le 2 \\ x^2 - 4, & \text{otherwise} \end{cases}$$

If $x \neq \pm 2$, then $f'(x) = 0 \iff x = 0$.

Critical points: $\boxed{-2, 0, 2.}$

Extreme Value Theorem: If f is a continuous function on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum. on $[a,b]$.

EXAMPLE 6. *Find the absolute extrema for* $f(x) = \underbrace{x^3 - 3x^2 + 3x}_{\text{continuous}}$ *on the interval* I *where*

(a) $I = [-1, 3]$ closed interval.

1) Find critical number of $f(x)$:

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0$

$x = 1$ critical number

2) Calculate value/s of f at critical point/s on I.

$x = 1$ is in $[-1, 3]$

$f(1) = 1^3 - 3\cdot 1^2 + 3\cdot 1 = \boxed{1}$

3) Calculate values of f at endpoints of I:

$f(-1) = (-1)^3 - 3\cdot(-1)^2 + 3\cdot(-1) = \boxed{-7}$

$f(3) = 3^3 - 3\cdot 3^2 + 3\cdot 3 = \boxed{9}$

4) Choose max and min from the obtained values of f:

$1, -7, 9$

$\max\limits_{[-1,3]} f(x) = f(3) = 9$ abs. max value

$\min\limits_{[-1,3]} f(x) = f(-1) = -7$ abs. min value.

(b) $I = [-1, 1]$

Candidates for abs. extremum:

$x = -1$ (as an endpoint)

$x = 1$ (as an endpoint and critical number).

$f(-1) = \boxed{-7}$ abs. max on $[-1,1]$: $f(1) = 1$:

$f(1) = \boxed{1}$ abs. min on $[-1,1]$ $f(-1) = -7$.

(c) $I = [-1, 0]$

Candidates for abs. extremum:

the endpoints only, because the critical point $x = 1$ is not on $[-1, 0]$.

$f(-1) = \boxed{-7}$ ← abs. min on $[-1,0]$

$f(0) = \boxed{0}$ ← abs. max on $[-1,0]$.