5.3: Derivatives and Shapes of Curves

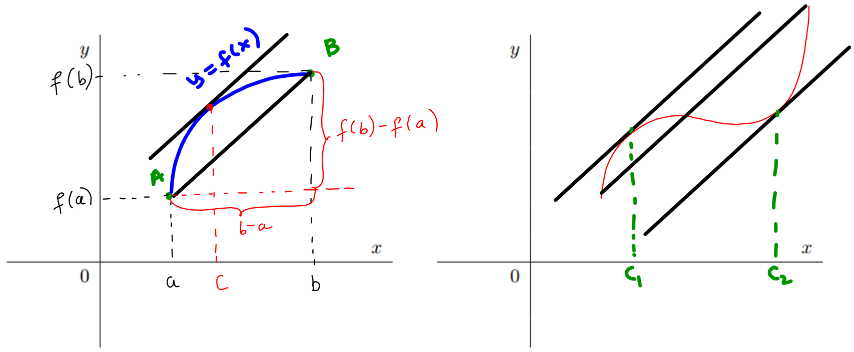
Mean Value Theorem: Suppose a function f is continuous on the (closed) interval $[a,b]$ and differentiable on the (open) interval (a,b). Then there is a number c such that $a<c<b$ and

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

or, equivalently,

$$f(b)-f(a)=f'(c)(b-a).$$

Illustration: $m_{AB}=\dfrac{f(b)-f(a)}{b-a}$



EXAMPLE 1. *Find a number c that satisfies the conclusion of the* MVT *Mean Value Theorem on the interval* $[0,2]$ *when* $f(x)=x^3+x-1$.

f(x) is a polynomial (i.e. it is continuous and differentiable everywhere)

In particular, f(x) is contin. on $[0,2]$ and diff. on $(0,2)$.

Hence, by MVT, there exists c such that $0<c<2$ and $f'(c)=\dfrac{f(2)-f(0)}{2-0}$

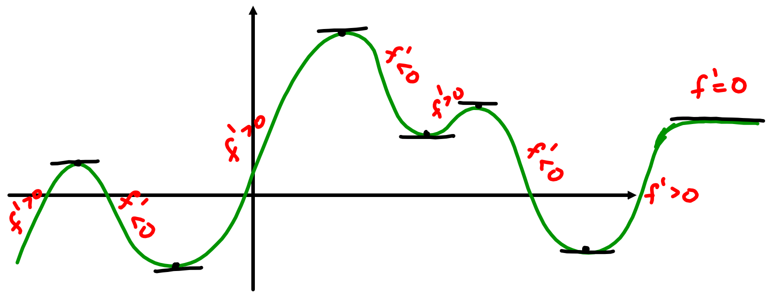
$f(x)=x^3+x-1$
$f(2)=8+2-1=9$
$f(0)=-1$
$f'(x)=3x^2+1$
$f'(c)=3c^2+1$

$$3c^2+1=\frac{9-(-1)}{2}$$
$$3c^2+1=5$$
$$3c^2=4$$
$$c^2=\frac{4}{3}$$
$$c=\pm\frac{2}{\sqrt{3}}$$

Since $0<c<2$, we get $\boxed{c=\frac{2}{\sqrt{3}}}$

Test for increasing/decreasing

- If $f'(x) > 0$ on an interval , then f is increasing on that interval.
- If $f'(x) < 0$ on an interval , then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval , then f is constant on that interval.



EXAMPLE 2. *Determine all intervals where the following function*

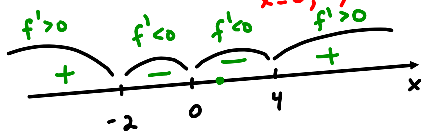
$$f(x) = x^5 - \frac{5}{2}x^4 - \frac{40}{3}x^3 - 12$$

is increasing or decreasing.

$f'(x) = 5x^4 - 10x^3 - 40x^2$

$f'(x) = 5x^2(x^2 - 2x - 8) = 5x^2(x-4)(x+2) = 0$

$x = 0, 4, -2$ critical numbers



$f(x)$ is increasing on $(-\infty, -2) \cup (4, \infty)$

$f(x)$ is decreasing on $(-2, 4)$.

First Derivative Test: Suppose that $x = c$ is a critical point of a continuous function f.

- If $f'(x)$ changes from negative to positive at $x = c$, then f has a local minimum at c.
- If $f'(x)$ changes from positive to negative at $x = c$, then f has a local maximum at c.
- If $f'(x)$ does not change sign at $x = c$, then f has no local maximum or minimum at c.

REMARK 3. The first derivative test only classifies critical points as local extrema and not as absolute extrema.

EXAMPLE 4. *For function from Example 2 identify all local extrema.*

Local max at $x=-2$

Local min at $x=4$

At $x=0$ no local extrema

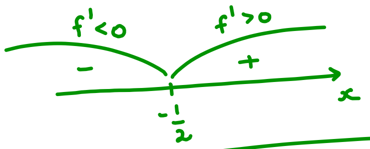
EXAMPLE 5. *Find all intervals of increase and decrease of* $f(x) = xe^{2x}$ *and identify all local extrema.*

$$f'(x) = (xe^{2x})' = e^{2x} + 2xe^{2x} = \underbrace{e^{2x}}_{>0}(1+2x) = 0$$

$\Downarrow$

$1+2x=0$

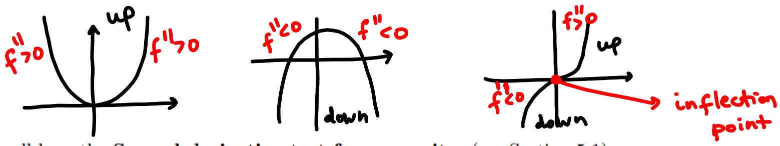
$x=-\frac{1}{2}$ critical number



$f(x)$ is increasing on $(-\frac{1}{2}, \infty)$

$f(x)$ is decreasing on $(-\infty, -\frac{1}{2})$

$f(x)$ has local min at $x=-\frac{1}{2}$



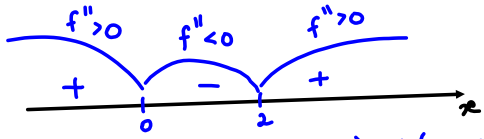
Recall here the **Second derivative test for concavity.** (see Section 5.1):

- If $f''(x) > 0$ for all x on an interval, then f is concave up on that interval.
- If $f''(x) < 0$ for all x on an interval, then f is concave down on that interval.

In addition, if f changes concavity at $x = a$, and $x = a$ is in the domain of f, then $x = a$ is an inflection point of f.

EXAMPLE 6. *Find intervals of concavity and inflection points of f, if $f'(x) = 4x^3 - 12x^2$.*

$$f''(x) = (f'(x))' = (4x^3 - 12x^2)' = 12x^2 - 24x = 12x(x-2)$$



$f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$

$f(x)$ is concave down on $(0, 2)$

inflection points are $x=0$ and $x=2$.

Second derivative test for local extrema: Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

REMARK 7. If $f'(c) = 0$ and $f''(c) = 0$ or does not exist, then the test fails. In the case $f''(c)$ does not exist we use the first derivative test to find the local extrema.

EXAMPLE 8. *Find the local extrema for $f(x) = 1 - 3x + 5x^2 - x^3$.*

$$f'(x) = -3 + 10x - 3x^2 = 0$$

$$3x^2 - 10x + 3 = 0$$

$$3(x-3)\left(x - \tfrac{1}{3}\right) = 0$$

$x = 3$, $x = \frac{1}{3}$ critical numbers

$$f''(x) = 10 - 6x$$

$f''(3) = 10 - 6 \cdot 3 < 0 \Rightarrow$ local max at $x=3$

$f''(\frac{1}{3}) = 10 - 6 \cdot \frac{1}{3} > 0 \Rightarrow$ local min at $x = \frac{1}{3}$.