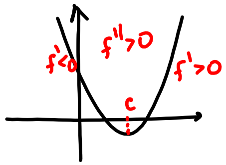


5.5: Applied Maximum and Minimum Problems

OPTIMIZATION PROBLEMS

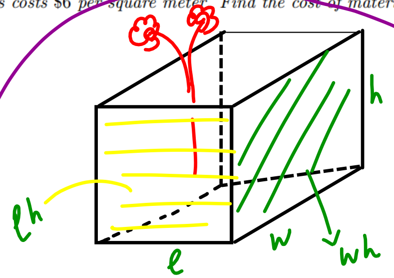
First derivative test for absolute extrema: Suppose that c is a critical number of a continuous function f defined on an interval.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Alternatively,

- *If $f''(x) < 0$ for all x (so f is always concave downward) then the local maximum at c must be an absolute maximum.*
- *If $f''(x) > 0$ for all x (so f is always concave upward) then the local minimum at c must be an absolute minimum.*

EXAMPLE 1. *A rectangular storage container with an open top is to have a volume of 10m^3. The length of its base is twice the width. Material for the base costs \$10 per squire meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.*



$V = lwh = 10$

$l = 2w$

Cost for base: $10\,lw$

Cost for sides: $6(2lh + 2wh)$
$= 12h(l+w)$

Total cost $= 10lw + 12h(l+w)$

$l = 2w$, $h = \frac{10}{lw} = \frac{10}{2w\cdot w} = \frac{5}{w^2}$

Cost function: $C(w) = 10(2w)w + \frac{12\cdot 5}{w^2}(2w+w)$

$C(w) = 20w^2 + \frac{60}{w^2}\cdot 3w$

$$\boxed{C(w) = 20\left(w^2 + \frac{9}{w}\right)}$$

Note that $w > 0$

Find absolute extrema for $C(w)$.

Critical points:

$$C'(w) = 20\left(2w - \frac{9}{w^2}\right) = 0$$

$$2w - \frac{9}{w^2} = 0$$

$$2w = \frac{9}{w^2}$$

$$w^3 = \frac{9}{2}$$

$$w^3 = \frac{9}{2}$$

$$w = \sqrt[3]{\frac{9}{2}}$$ critical point.

Find $C''(w)$:

$$C''(w) = \left(20\left(2w - \frac{9}{w^2}\right)\right)'$$

$$= 20\left(2 + \frac{2\cdot 9}{w^3}\right) > 0$$

for all $w > 0$

So, $C(w)$ has absolute min on $(0, \infty)$.

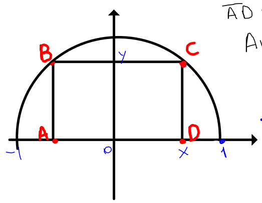
It remains to find the abs. min value

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(w^2 + \frac{9}{w}\right)\Bigg|_{w=\sqrt[3]{\frac{9}{2}}}$$

$$= \frac{20\,(w^3 + 9)}{w}\Bigg|_{w=\sqrt[3]{\frac{9}{2}}}$$

$$= \frac{20\left(\frac{9}{2} + 9\right)}{\sqrt[3]{\frac{9}{2}}} = \$\ldots$$

EXAMPLE 2. *A rectangle is bounded by the x-axis and the semicircle $y=\sqrt{1-x^2}$. What length and width should the rectangle have so that its area is a maximum?* (Equivalently,find the dimensions of the largest rectangle that can be inscribed in the semi-disk with radius 1.)



$\overline{AB}=\overline{CD}=y$

$\overline{AD}=\overline{BC}=2x$

Area of ABCD:

Area $=\overline{AB}\cdot\overline{AD}=2xy$

But $y=\sqrt{1-x^2}$. Hence

Area $=\boxed{2x\sqrt{1-x^2}=A(x)}$

Find absolute max of $A(x)$ when $0<x<1$

Critical points: $A'(x)=0$

$$A'(x)=2\left[\sqrt{1-x^2}+\frac{x(-2x)}{2\sqrt{1-x^2}}\right]$$

$$=2\left[\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}\right]=$$

$$=\frac{2[(\sqrt{1-x^2})^2-x^2]}{\sqrt{1-x^2}}=\frac{2[1-x^2-x^2]}{\sqrt{1-x^2}}$$

So, $A'(x)=\dfrac{2(1-2x^2)}{\sqrt{1-x^2}}=0 \Rightarrow 1-2x^2=0$

$x=\pm\frac{1}{\sqrt{2}}$.

$A(x)$ has the following critical points: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, -1$ (at last two points $A'(x)$ does not exist)

However, only critical point $x=\frac{1}{\sqrt{2}}$ satisfy the condition $0<x<1$.

Note that $A'(x)>0$ if $0<x<\frac{1}{\sqrt{2}}$

and $A'(x)<0$ if $\frac{1}{\sqrt{2}}<x<1$.

Thus, by the First derivative test for absolute extrema, $A(x)$ has the absolute maximum at $x=\frac{1}{\sqrt{2}}$.

Conclusion:

Dimensions of the rectangle with maximal area are:

length : $2x=2\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$

width : $y=\sqrt{1-x^2}=\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$.