Section 5.7: Antiderivatives

DEFINITION 1. *A function F is called an* **antiderivative** *of f on an interval I if $F'(x) = f(x)$ for all x in I.*

EXAMPLE 2. **(a)** *Is the function $F(x) = x\ln(x) - x + \sin x$ is an antiderivative of $f(x) = \ln(x) + \cos x$?*

$$F'(x) = (x\ln x - x + \sin x)' = \ln x + x \cdot \frac{1}{x} - 1 + \cos x$$

$$= \ln x + 1 - 1 + \cos x = \ln x + \cos x = f(x)$$

YES

(b) *Is the function $F(x) = x\ln(x) - x + \sin x + 10$ is an antiderivative of $f(x) = \ln(x) + \cos x$?*

YES

(c) *What is the most general antiderivative of $f(x) = \ln(x) + \cos x$?*

$$F(x) = x\ln x - x + \sin x + \underbrace{C}_{\text{a constant}}$$

$F' = f$

THEOREM 3. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.*

EXAMPLE 4. *Find the most general antiderivative of $f(x) = 2x$.*

$$f(x) = 2x$$

$$F(x) = x^2 + C$$

Table of Antidifferentiation Formulas

$F'(x) = f(x)$

Function f	Particular antiderivative F	Most general antiderivative
$k \quad (k \in \mathbb{R})$	kx	$kx + C$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	
e^x	e^x	
$\cos x$	$\sin x$	$+C$
$\sin x$	$-\cos x$	
$\sec^2 x$	$\tan x$	
$\csc^2 x$	$-\cot x$	
$\sec x \tan x$	$\sec x$	
$\csc x \cot x$	$\csc x$	
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	
$\dfrac{1}{1+x^2}$	$\arctan x$	

$$(x^{n+1})' = (n+1)\,x^n$$

$$x^n = \frac{(x^{n+1})'}{n+1}$$

$$x^n = \left(\frac{x^{n+1}}{n+1}\right)'$$

EXAMPLE 5. *Find the most general antiderivative of f where*

(a) $f(x) = 5\sin x + \sqrt[8]{x^7} + 15e^x - 17$

$= 5(\sin x) + x^{\frac{7}{8}} + 15(e^x) + (-17)$

$F(x) = 5(-\cos x) + \dfrac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + 15e^x - 17x + C$

$= -5\cos x + \frac{8}{15}x^{\frac{15}{8}} + 15e^x - 17x + C$

(b) $f(x) = \dfrac{3x + 8 - x^2}{x^3} = \dfrac{3x}{x^3} + \dfrac{8}{x^3} - \dfrac{x^2}{x^3}$

$= 3x^{-2} + 8x^{-3} - \dfrac{1}{x}$

$F(x) = \dfrac{3x^{-2+1}}{-2+1} + \dfrac{8x^{-3+1}}{-3+1} - \ln|x| + C$

$= \dfrac{-3}{x} - \dfrac{4}{x^2} - \ln|x| + C$

(c) $f(x) = e^x + (1 - x^2)^{-1/2}$

$= e^x + \dfrac{1}{\sqrt{1-x^2}}$

$F(x) = e^x + \arcsin(x)$

EXAMPLE 6. *Find $f(x)$ given that $f'(x) = 4 - 3(1+x^2)^{-1}$, $f(1) = 0$.*

$$f'(x) = 4 - \frac{3}{1+x^2}$$

Antiderivative of $f'(x)$ is

$$f(x) = 4x - 3\arctan(x) + C$$

But $f(1)=0$, so

$$f(1) = 4\cdot 1 - 3\arctan(1) + C = 0$$

$$4 - 3\frac{\pi}{4} + C = 0$$

$$C = \frac{3\pi}{4} - 4$$

$$\boxed{f(x) = 4x - 3\arctan(x) + \frac{3\pi}{4} - 4}$$

EXAMPLE 7. *A particle is moving according to acceleration* $a(t) = 2t + 2$. *Find the position,* $s(t)$, *of the object at time* t *if we know* $s(0) = 1$ *and* $v(0) = -2$.

$$v'(t) = a(t) = 2t + 2$$

$$v(t) = t^2 + 2t + C$$

But $v(0) = -2$, so $v(0) = 0^2 + 2 \cdot 0 + C = -2$

$$C = -2$$

$$s'(t) = v(t) = t^2 + 2t - 2$$

$$s(t) = \frac{t^3}{3} + t^2 - 2t + C_1$$

But $s(0) = 1$, so $s(0) = \frac{0^3}{3} + 0^2 - 2 \cdot 0 + C_1 = 1$

$$C_1 = 1$$

$$\boxed{s(t) = \frac{t^3}{3} + t^2 - 2t + 1}$$