9 FUNCTIONS

9.1 The Definition of Function

DEFINITION 1. *Let X and Y be nonempty sets. A* **function** *f from the set X to the set Y is a correspondence that assigns to each element x in the set X one and only one element y in the set Y, which is denoted by $f(x)$.*

We call X the **domain** *of f and Y the* **codomain** *of f.*

If $x \in X$ and $y \in Y$ are such that $y = f(x)$, then y is called the **value** of f at x, or the **image** of x under f. We may also say that f **maps** x to y.

Using diagram

DEFINITION 2. *Two functions f and g are* **equal** *if they have the same domain and the same codomain and if $f(x) = g(x)$ for all x in domain.*

DEFINITION 3. *The* **graph** *of $f : X \to Y$ is the set*

$$G_f = \{(x, y) \in X \times Y | y = f(x)\}.$$

- We can determine a function from its domain, codomain, and graph.
- We can describe a function by formula, by listing its values, or by words.

EXAMPLE 4. *Let $X = \{2, 4, 6\}$ and $Y = \{a, b, c, d\}$. Determine in which of the following cases, f is function from X to Y.*

(a) $f(2) = b, f(4) = a, f(6) = d$

(b) $f(2) = c, f(4) = c, f(6) = c$

(c) $f(2) = a, f(4) = b, f(6) = c, f(4) = d$

(d) $f(2) = c, f(6) = d$

Some common functions

- **Identity** function $I_X : X \to X$ maps every element to itself:

- **Polynomial** of degree n with real coefficients $a_0, a_1, \ldots, a_n$ is a function from $\mathbb{R}$ to $\mathbb{R}$

  Polynomials of degrees 0,1,2,3 are constant, linear, quadratic, cubic, respectively.

EXAMPLE 5. *Let $f : X \to Y$ be defined by $f(x) = x^3 + 3$. In each of the following cases find its graph and illustrate it .*

(a) $X = Y = \mathbb{R}$

(b) $X = \{-1, 0, 1\}\,,\ Y = \mathbb{R}$

Range (or Image) of a Function

DEFINITION 6. *Let $f : X \to Y$ be a function. The* **range** *of f (also called the* ***image*** *of f) is the set*

$$\{y \in Y | y = f(x) \quad for \quad some \quad x \in X\}.$$

We denote the range (or image) of the function f by ran$f$ (or Imf).

EXAMPLE 7. *Let $f : X \to Y$ be a function. Using symbols complete the following*

- $\text{ran} f \subseteq$ ___________
- $\forall y \in Y, y \in \text{ran} f \;\; \Leftrightarrow$ ________________________________
- $y \notin \text{ran} f \;\; \Leftrightarrow$ ________________________________

EXAMPLE 8. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \cos x$. Find* ran$f$.

EXAMPLE 9. *Let $f : [\frac{1}{3}, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{3x - 1}$ and $S = \{y \in \mathbb{R} | \; y \geq 0\}$. Prove that* ran$f = S$.

9.5 Composition of Functions

DEFINITION 10. *Let A, B, and C be nonempty sets, and let $f : A \to B$, $g : B \to C$. We define a function*

$$g \circ f : A \to C,$$

called the **composition** *of f and g, by*

$$(g \circ f)(a) =$$

Using diagram

EXAMPLE 11. *Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{r, s, t, u, v\}$ and define the functions $f : A \to B$, $g : B \to C$ by their graphs:*

$$G_f = \{(1, b), (2, d), (3, a), (4, a)\}, \qquad G_g = \{(a, u), (b, r), (c, r), (d, s)\}.$$

Find $g \circ f$. What about $f \circ g$?

EXAMPLE 12. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$ and $g(x) = x \sin x$. Find $f \circ g$ and $g \circ f$.*

PROPOSITION 13. *Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

i.e. composition of functions is associative.

Proof.

Section 9.3 Surjective (or onto) and Injective (or one-to-one) Functions

Surjective functions ("onto")

DEFINITION 14. *Let $f : X \to Y$ be a function. Then f is* **surjective** *(or a surjection) if the range of f coincides with its codomain, i.e.*

$$\text{ran} f = Y.$$

Note: surjection is also called "onto".

`Proving surjection:`

We know that for all $f : X \to Y$: ________________________

Thus, *to show that $f : X \to Y$ is a surjection it is sufficient to prove that* ________________

In other words,
*to prove that $f : X \to Y$ is a surjective function it is sufficient to show that*____

Question: How to disprove surjectivity?

EXAMPLE 15. *Let* $f:\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to[0,\infty)$ *defined by* $f(x)=g(x)=x^4$. *Determine whether the following are true*

(a) $\text{ran} f=\text{ran}(g)$

(b) $f=g$

(c) f *is surjective*

(d) g *is surjective*

EXAMPLE 16. *Prove that the function* $f:\mathbb{R}-\{2\}\to\mathbb{R}-\{3\}$ *defined by* $f(x)=\dfrac{3x}{x-2}$ *is surjective.*

Injective functions ("one to one")

DEFINITION 17. *Let* $f:X\to Y$ *be a function. Then* f *is* **injective** *(or an injection) if whenever* $x_1,x_2\in X$ *and* $x_1\neq x_2$, *we have* $f(x_1)\neq f(x_2)$.

In other words, f *is injective if and only if the ranges of every two distinct points in the domain of* f *are distinct.*

EXAMPLE 18. *Given $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$. Determine whether the following functions are injective. Justify your answer.*

(a) $f : X \to Y$ *defined by* $G_f = \{(1, 3), (2, 4), (3, 5)\}$

(b) $g : X \to Y$ *defined by* $G_g = \{(1, 5), (2, 4), (3, 4)\}$

`Proving injection:`
Let $P(x_1, x_2) : x_1 \neq x_2$ and $Q(x_1, x_2) : f(x_1) \neq f(x_2)$.

Then by definition f *is injective if* ____________________.

Using contrapositive, we have ______________________________________.

In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 19. *Prove or disprove injectivity of the following functions.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[5]{x}$.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$.

(c) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n/2 & \text{if } n \in \mathbb{E}, \\ 2n & \text{if } n \in \mathbb{O}. \end{cases}$

(d) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n & \text{if } n \in \mathbb{E}, \\ 5n & \text{if } n \in \mathbb{O}. \end{cases}$

`Discussion Exercise.`

- Must a strictly increasing or decreasing function be injective?

- Must an injective function be strictly increasing or decreasing?

EXAMPLE 20. *Prove or disprove injectivity of the following functions. In each case,* $f : \mathbb{R} \to \mathbb{R}$.

(a) $f(x) = 3x^5 + 5x^3 + 2x + \pi$.

(b) $f(x) = x^{12} + x^8 - x^4 + 12$.

9.4 Bijective functions

DEFINITION 21. *A function that is both surjective and injective is called* **bijective** *(or bijection.)*

Complete the following.

- f is not bijective $\Leftrightarrow$ ____

- f is surjective $\Leftrightarrow$ (codom$f \subseteq$ __________) $\Leftrightarrow$ ($\forall y, y \in$ codom$f \Rightarrow$ __________) $\Leftrightarrow$

  $\Leftrightarrow$ ($\forall y, y \in$ codom$f \Rightarrow \exists x \in$ dom$f$____________) $\Leftrightarrow$ ($\forall y \in$ codom$f, \exists x \in$ dom$f$__________)

  In other words, f *is surjective if and only if every point in* codom$f$ *has a preimage in the* dom$f$.

  If in addition f is injective, then we obtain

- f is bijective $\Leftrightarrow$ ($\exists!\ x \in$ dom$f$__________)

  In other words, f *is bijective if and only if every point in* codom$f$ *has a unique preimage in the* dom$f$.

EXAMPLE 22. *Determine which of the following functions are bijective.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. ***(b)*** $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

PROPOSITION 23. *Let $f : A \to B$ and $g : B \to C$. Then*

i. *If f and g are surjections, then $g \circ f$ is also a surjection.*

`Proof.`

ii. *If f and g are injections, then $g \circ f$ is also an injection.*

`Proof.`

COROLLARY 24. *If f and g are bijections, then $g \circ f$ is also a bijection.*

PROPOSITION 25. *Let $f : X \to Y$. Then $f \circ I_X = f$ and $I_Y \circ f = f$.*

9.6 Inverse Functions

DEFINITION 26. *Let $f : X \to Y$ be a function. We say that f is* **invertible** *if there is a function $g : Y \to X$ such that for all $x \in X$ and for all $y \in Y$,*

$$y = f(x) \quad \Leftrightarrow \quad x = g(y).$$

We say that such a function g is an ***inverse function*** *of f.*

Question: *What is the inverse of g?*

REMARK 27. f is invertible if and only if its inverse is invertible.

EXAMPLE 28. *Show that the function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ defined by $f(x) = \dfrac{3x}{x-2}$ is invertible and find its inverse function. (Note that the given function is bijective.)*

PROPOSITION 29. *A function $f : X \to Y$ is invertible if and only if there exists a function $g : Y \to X$ such that*

$$g \circ f = I_X \quad \text{and} \quad f \circ g = I_Y.$$

PROPOSITION 30. *The inverse function is unique.*

`Proof.`

Notation

When $f : X \to Y$ is invertible, the unique inverse function is denoted by f^{-1}, and $f^{-1} : Y'X$.

REMARK 31. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) =$ ________

The function $f(x) = 3x^5 + 5x^3 + 2x + 220$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 32. *A function $f : A \to B$ is invertible if and only if f is bijective.*

COROLLARY 33. *If a function $f : A \to B$ is bijective, so is f^{-1}.*