

## 9.1 The Definition of Function (continued)

### FUNCTIONS AND SETS

#### Image of a Set

DEFINITION 1. Let  $f : X \rightarrow Y$  be a function. If  $A \subseteq X$ , we define  $f(A)$ , the **image** of  $A$  under  $f$ , by

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

EXAMPLE 2. Let  $f : X \rightarrow Y$ . Complete:

- (a) If  $A \subseteq X$  then  $f(A) \subseteq \underline{\hspace{2cm}} \subseteq \underline{\hspace{2cm}}$
- (b)  $f(X) = \underline{\hspace{2cm}}$
- (c)  $y \in f(A) \Leftrightarrow \underline{\hspace{4cm}}$
- (d)  $y \notin f(A) \Leftrightarrow \underline{\hspace{4cm}}$

EXAMPLE 3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \cos x$ . Find the following

- (a)  $f([- \pi/2, \pi/2])$
- (b)  $f([- \pi/2, 0])$
- (c)  $f([- \pi/2, \pi/2017])$
- (d)  $f([- \pi/4, \pi/4])$
- (e) Does  $-\frac{1}{2017}$  belong to  $f([- \pi/2, \pi/2])$ ?

**Inverse Image**

DEFINITION 4. Let  $f : X \rightarrow Y$  be a function and let  $B$  be a subset of its codomain (i.e.  $B \subseteq Y$ ). Then the **inverse image** of  $B$  (written  $f^{-1}(B)$ ) is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

The inverse image  $f^{-1}(B)$  is a subset of domain of  $f$  containing all preimages of points from  $B$  under  $f$ .

EXAMPLE 5. Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . Complete:

(a)  $f^{-1}(B) \subseteq$  \_\_\_\_\_

(b)  $x \in f^{-1}(B) \Leftrightarrow$  \_\_\_\_\_

(c)  $x \notin f^{-1}(B) \Leftrightarrow$  \_\_\_\_\_

(d) If  $x \in X$  and  $y \in Y$ , then

$$f(x) \text{ _____ } Y, \quad f(\{x\}) \text{ _____ } Y, \quad f^{-1}(y) \text{ _____ } X, \quad f^{-1}(\{y\}) \text{ _____ } X,$$

EXAMPLE 6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^4$ . Find the following:

(a)  $f([1, 2]) =$

(b)  $f([-2, -1]) =$

(c)  $f^{-1}([1, 16]) =$

(d)  $f^{-1}([-16, -1]) =$

(e)  $f^{-1}([-1, 1]) =$

(f)  $f^{-1}(\mathbb{R}) =$

(g)  $f^{-1}(\mathbb{R}^+) =$

EXAMPLE 7. Let  $A \subseteq \text{dom}(f)$  and  $B \subset \text{codom}(f)$ . Determine the truth or falsehood of the following

(a)  $f^{-1}(f(A)) = A$

(b)  $f(f^{-1}(B)) = B$

EXAMPLE 8. Let  $A_1, A_2 \subseteq \text{dom}(f)$ . Determine the truth or falsehood of the following

$$f(A_1) \cap f(A_2) = f(A_1 \cap A_2)$$

**Summary**

Let  $f : X \rightarrow Y$  and  $A \subseteq X$  and  $B \subseteq Y$ . Then above definitions imply the following tautologies

- $\forall y \in Y, ((y \in \text{ran}(f)) \Leftrightarrow (\exists x \in X \ni f(x) = y))$ .
- $\forall y \in Y, ((y \in f(A)) \Leftrightarrow (\exists x \in A \ni f(x) = y))$ .
- $\forall x \in X, ((x \in f^{-1}(B)) \Leftrightarrow (f(x) \in B))$ .

Also note that

- If  $B \subseteq \text{ran}(f)$  then  $(S = f^{-1}(B)) \Rightarrow (f(S) = B)$ .

**EXAMPLE 9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 3x + 4$  and let  $B = \{x \in \mathbb{R} | x > 0\}$ . Find  $f^{-1}(B)$ . Justify your answer (give a formal proof).

**EXAMPLE 10.**  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = 5x - 4$ . Find  $f([0, 1])$ . Justify your answer (give a formal proof).

PROPOSITION 11. Let  $f : X \rightarrow Y$ . If  $A_1 \subseteq A_2 \subseteq X$  then  $f(A_1) \subseteq f(A_2)$ .

*Proof.*

PROPOSITION 12. Let  $f : X \rightarrow Y$ . If  $B_1$  and  $B_2$  are subsets of  $Y$  then

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

*Proof.*

EXAMPLE 13. (cf. Example 8.) Let  $f : X \rightarrow Y$  and  $A_1$  and  $A_2$  be subsets of  $X$ .

(a) Prove that  $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ .

(b) Prove that if, in addition,  $f$  is an injective function, then  $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ .

EXAMPLE 14. (cf. Example 7.) Let  $f : X \rightarrow Y$  and  $B$  be a subset of  $Y$ .

(a) Prove that  $f(f^{-1}(B)) \subseteq B$ .

(b) Prove that if, in addition,  $f$  is an surjective function, then  $f(f^{-1}(B)) = B$ .