1 Mathematical Reasoning

Prove: "*For every odd integer n, the integer 3n+7 is even.*"

1.1 Statements

DEFINITION 1. *A statement is any declarative sentence that is either true or false.*

A statement cannot be neither true nor false and it cannot be both true and false.

1. The integer 5 is odd.
2. The integer 24 is prime.
3. $15 + 7 = 22$
4. Apple manufactures computers.
5. Apple manufactures the world's best computers.
6. Did you buy IBM?
7. I am telling a lie.
8. What happen when Pinocchio says: "My nose will grow now"?

   An open sentence is any declarative sentence containing one or more variables that is not a statement but becomes a statement when the variables are assigned values.
9. $x + 5 = 7$
10. He is a student.
11. $x^2 + y^2 = 1$

An open sentence can be made into a statement by using **quantifiers**.

Universal: $\forall x$ means for all/for every assigned value a of x.
Existential: $\exists x$ means that for some assigned vales a of x.

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. The variable that is not bound is called a **free** variable.

12. For every real number x, $x + 5 = 7$.
Quantifiers:

13. The area of a rectangle is its length times its width.
Quantifiers:

14. A triangle may be equilateral.
Quantifiers:

15. $15 - 5 = 10$
Quantifiers:

16. The sum of an even integer and an odd integer is even.
Quantifiers:

17. All positive real numbers have a square root.
Quantifiers:

18. A real-valued function that is continuous at 0 is not necessarily differentiable at 0.
Quantifiers:

NEGATIONS

DEFINITION 2. *If P is a statement, then the* **negation** *of P, written $\neg P$ (read "not P"), is the statement "P is false".*

19. All continuous functions are differentiable.

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20. P: Barack Obama won the Nobel Peace Prize.

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21. $P: \quad 5^3 = 120 \qquad \neg P:$______________

22. $P(x): \quad x^2 + x + 1 = 0 \qquad \neg P(x):$______________

23. $P(x, y): \quad x^4 + y^4 = 0 \qquad \neg P(x, y):$______________

24. P : If n is an odd integer then $3n + 7$ is odd.

    $\neg P$____

25. P : Every car on the parking lot #47 was with valid permit.

    $\neg P$____

26. P :There exist real numbers a and b such that $(a + b)^2 = a^2 + b^2$.

    $\neg P$____

Rules to negate statements with quantifiers:

27. P : For every even integer n there exists an integer m such that $n = 2m$.

    $\neg P$ ____

28. P : There exists a positive integer n such that $m(n+5) < 1$ for every integer m.

    $\neg P$ ____

EXAMPLE 3. *P: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*
Question: Is the following "proof" valid?
Let $n = 6$. Then $n^2 = 6^2 = 36$ and 36 is a multiple of 4, but 6 is not a multiple of 4. Therefore, the statement P is FALSE.□

1.2 Compound Statements

1. P: Some math tests are long.

2. Q: Some math tests are difficult.

Logical connectivity	write	read	meaning
Conjunction	$P \wedge Q$	P and Q	Both P and Q are true
Disjunction	$P \vee Q$	P or Q	P is true or Q is true

TRUTH TABLES

P: Ben is a student.
Q: Ben is a teaching assistant.

P	Q	$P \wedge Q$	$Q \wedge P$	$P \vee Q$	$Q \vee P$

EXAMPLE 4. *Rewrite the following open statements using disjunction or conjunction.*

(a) $P(x): \quad |x| \geq 10.$

(b) $P(x): \quad |x| < 10.$

Two compound statements are **logically equivalent** (write "$\equiv$") if they have the same truth tables, which means they both are true or both are false.

EXAMPLE 5. *Let P and Q be statement forms. Determine whether the compound statements $\neg P \wedge Q$ and $\neg P \vee Q$ are logically equivalent (i.e. both true or both false).*

P	Q			

Some Fundamental Properties of Logical Equivalence

THEOREM 6. *For the statement forms P, Q and R,*

- $\neg(\neg P) \equiv$
- *Commutative Laws*
  $P \vee Q \equiv$
  $P \wedge Q \equiv$
- *Associative Laws*
  $P \vee (Q \vee R) \equiv$
  $P \wedge (Q \wedge R) \equiv$
- *Distributive Laws*
  $P \vee (Q \wedge R) \equiv$
  $P \wedge (Q \vee R) \equiv$
- *De Morgan's Laws*
  $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
  $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

Proof. Each part of the theorem is verified by means of a truth table.

P	Q					

EXAMPLE 7. *Rewrite the rules to negate statements with quantifiers in terms of logical equivalence:*

$\neg(\forall x, P(x)) \equiv$

$\neg(\exists x \ni P(x)) \equiv$

$\neg(\forall x, (P(x) \vee Q(x)) \equiv$

$\neg(\forall x, (P(x) \wedge Q(x)) \equiv$

$\neg(\exists x \ni (P(x) \vee Q(x)) \equiv$

$\neg(\exists x \ni (P(x) \wedge Q(x)) \equiv$

EXAMPLE 8. *Negate:*
P*: There exists a prime number* p *which is greater then 7 and less than 10.*

$\neg P$____

Tautologies and Contradictions

Tautology: statement that is always true
Contradiction: statement that is always false

P	$\neg P$	$P \vee (\neg P)$	$P \wedge (\neg P)$
T			
F			

1.3 Implications

DEFINITION 9. *Let* P *and* Q *be statements. The* **implication** $P \Rightarrow Q$ *(read "*P *implies* Q*") is the statement "If* P *is true, then* Q *is true."*

EXAMPLE 10. *If* n *is odd, then* $3n + 7$ *is even.*

The truth table for implication:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

EXAMPLE 11. P : *You earn an A on the final exam.*
Q*: You get an A for your final grade.*

$P \Rightarrow Q$

`Different ways of expressing` $P \Rightarrow Q$:

If P is true, then Q is true
Q is true if P is true
P implies Q
P is true only if Q is true
P is sufficient for Q
Q is necessary for P.

EXAMPLE 12. *For a triangle T, let*

$P(T)$: *T is equilateral* $Q(T)$: *T is isosceles.*

State $P(T) \Rightarrow Q(T)$ in a variety of ways:

EXAMPLE 13. P : *The function $f(x) = \sin x$ is differentiable everywhere.*
Q: *The function $f(x) = \sin x$ is continuous everywhere.*

$P \Rightarrow Q$ $Q \Rightarrow P$

`Proving Statements Containing Implications.`

Properties of Integers:

FACT 1 *The negative of every integer is an integer.*

FACT 2 *The sum (and difference) of every two integers is an integer.*

FACT 3 *The product of every two integers is an integer.*

FACT 4 *Every integer is either even, or odd.*

- **DIRECT PROOF**
  - Assume that P is true.
  - Draw out consequences of P.
  - Use these consequences to prove Q is true.

EXAMPLE 14. *If n is an even integer, then $5n^5$ is an even integer.*

EXAMPLE 15. Evaluate the proposed proof of the following result:

If a is an even integer and b is an odd integer, then $3a - 5b$ is odd.

Proof. Let a be an even integer and b be an odd integer. Then $a = 2n$ and $b = 2n + 1$ for some integer n. Therefore,

$$3a - 5b = 3(2n) - 5(2n + 1) = 6n - 10n - 5 = -4n - 5 = 2(-2n - 2) - 1.$$

Since $-2n - 2$ is an integer, $3a - 5b$ is odd. □

EXAMPLE 16. *The sum of every two odd integers is even.*

EXAMPLE 17. *Let x be an integer. If $5x - 7$ is odd, then $9x + 2$ is even.*

- **PROOF BY CASES**

  EXAMPLE 18. *If n is an integer, then $n^2 + 3n + 4$ is an even integer.*

  Hint: Use the following fact: "*Every integer number is either even. or odd.*"

`Negating an implication:` **Counterexamples**

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$$

REMARK 19. The negation of an implication is not an implication!

EXAMPLE 20. *Negate the statement: "For all $x, P(x) \Rightarrow Q(x)$."*

The value assigned to the variable x that makes $P(x)$ true and $Q(x)$ false is called a **counterexample** to the statement "For all $x, P(x) \Rightarrow Q(x)$."

EXAMPLE 21. Disprove the following statement:

If a real-valued function is continuous at some point, then this function is differentiable there.

Necessary and Sufficient Conditions

$P \Rightarrow Q$ can be expressed as

P is sufficient for Q.
or
Q is necessary for P.

Equivalently,

In order for Q to be true it is sufficient that P be true.
or
Q must be true in order to P to be true.

EXAMPLE 22. *Consider the following open sentences*
$P(x)$: *x is a multiple of 4.* $Q(x)$: *x is even. Complete:*

- *"$\forall x, P(x) \Rightarrow Q(x)$" is ____________________.*
- *$P(x)$ is a ____________________condition for Q to be true.*
- *$Q(x)$ is a____________________ condition for $P(x)$ to be true.*
- *$Q(x)$ is not a ____________________ condition for $P(x)$ to be true.*

EXAMPLE 23. *Consider the following open sentences*
$P(f)$: f is a differentiable function.
$Q(f)$: f is a continuous function.
Complete:

- *"$\forall f, P(f) \Rightarrow Q(f)$" is ____________________.*
- *"$\forall f, Q(f) \Rightarrow P(f)$" is ____________________.*
- *$Q(f)$ is a ____________________condition for f to be differentiable, but not a ____________________ condition.*
- *$P(f)$ is a ____________________ condition for f to be continuous.*

REMARK 24. Note however, if $P \Rightarrow Q$ is true, then it is not necessary that P is true in order for Q to be true. Even if Q is true, P may be false.

1.4 Contrapositive and Converse

Contrapositive

DEFINITION 25. *The statement* $\neg Q \Rightarrow \neg P$ *is called the* **contrapositive** *of the statement* $P \Rightarrow Q$.

EXAMPLE 26. *Let* P *and* Q *be statement forms. Prove that* $\neg Q \Rightarrow \neg P$ *is logically equivalent to* $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

Methods to prove an implication $P \Rightarrow Q$ (continued)

- **CONTRAPOSITIVE PROOF** (based on the equivalence $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$)
  - Assume that $\neg Q$ is true.
  - Draw out consequences of $\neg Q$.
  - Use these consequences to prove $\neg P$ is true.
  - It follows that $P \Rightarrow Q$.

  REMARK 27. If you use a contrapositive method, you must declare it in the beginning and then state **what is sufficient to proof.**

EXAMPLE 28. *Let* x *be an integer. If* $5x - 7$ *is even, then* x *is odd.*

Converse

DEFINITION 29. *The statement $Q \Rightarrow P$ is called a* **converse** *of the statement $P \Rightarrow Q$.*

Question: Are the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ logically equivalent? __________

EXAMPLE 30. *If m and n are odd integers then $m+n$ is even.*

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Biconditional "$\Leftrightarrow$"

DEFINITION 31. *The statement $P \Leftrightarrow Q$ (or P iff Q) is the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.*

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

THEOREM 32. *Let $\vec{a}$ and $\vec{b}$ be two non zero vectors. Then $\vec{a}$ is orthogonal to $\vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$.*

THEOREM 33. *Let n be an integer. Then n is even if and only if n^2 is even.*

Proof.

REMARK 34. $(P \Leftrightarrow Q) \equiv (\neg P \Leftrightarrow \neg Q)$

COROLLARY 35. *Let n be an integer. Then n is odd iff n^2 is odd.*

Methods to prove an implication $P \Rightarrow Q$ (continued)

EXAMPLE 36. *Let S and C be statement forms. Prove that $\neg S \Rightarrow (C \wedge \neg C)$ is logically equivalent to S.*

- **PROOF BY CONTRADICTION**
  - *Assume that P is true.*
  - *To derive a contradiction, assume that $\neg Q$ is true.*
  - *Prove a false statement C, using negation $\neg(P \Rightarrow Q) \equiv (P \wedge \neg Q)$.*
  - *Prove $\neg C$. It follows that Q is true. (The statement $C \wedge \neg C$ must be false, i.e. a contradiction.)*

PROPOSITION 37. *If m and n are integers, then $m^2 - 4n \neq 2$.*

Proof.

PROPOSITION 38. *Let be* $a, b,$ *and* c *be integers. If* $a^2 + b^2 = c^2$ *then* a *or* b *is an even integer.*

Proof.

DEFINITION 39. *A real number x is* **rational** *if $x = \frac{m}{n}$ for some integer numbers m and n. Also, x is* **irrational** *if it is not rational, that is*

PROPOSITION 40. *The number $\sqrt{2}$ is irrational.*