Mathematical Reasoning (Part I)[1]

Statements

DEFINITION 1. *A statement is any declarative sentence[2] that is either true or false, but not both.*

A statement cannot be neither true nor false and it cannot be both true and false.

1. The integer 5 is odd.
2. The integer 24277151704311 is prime.
3. $15 + 7 = 22$
4. Substitute the number 7 for x.
5. What is the derivative of $\cos x$?
6. Apple manufactures computers.
7. $|x| > 7$
8. The absolute value of a real number x is greater than 7.
9. The absolute value of a real number x is greater than 7.
10. $x^2 \geq 0$.
11. I am telling a lie.
12. What happen when Pinocchio says: "My nose will grow now"?

- **Set Terminology and Notation (very short introduction[3])**

  DEFINITIONS:
  Set is a well-defined collection of objects.
  Elements are objects or members of the set.

  - **Roster notation:**

    $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

  - **Indicating a pattern:**

    $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

[1]This part is covered in Sections 1.1-1.3 in the textbook.
[2]i.e. it has both a subject and a verb
[3]We will study SETS in Chapter 2!

Some Number sets:

- $\mathbf{R}$ is the set of all *real* numbers;
- $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of all *integers*;
- $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all *positive integers*;
- $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all *natural* numbers;
- $\mathbf{E}$ is the set of all *even integers*;
- $\mathbf{O}$ is the set of all *odd integers*;
- $n\mathbf{Z}$ is the set of all integers multiples of n ($n \in \mathbf{Z}$);

An **open sentence** *is any declarative sentence containing one or more variables, each variable representing a value in some prescribing set, called the* **domain** *of the variable, and which becomes a statement when values from their respective domains are substituted for these variables.*

1. $P(x) : x + 5 = 7$
2. $P(n) : n$ is divisible by 6.

EXAMPLE 2. *Discuss* $P(x) : (x-3)^2 \le 1$ *over* $\mathbf{Z}$.

EXAMPLE 3. *Discuss* $P(x, y) : x^2 + y^2 = 1$ *when* $x, y \in \mathbf{R}$.

The NEGATION of a Statement

DEFINITION 4. *If P is a statement, then the* **negation** *of P, written $\neg P$ (read "not P"), is the statement "P is false".*

Although $\neg P$ could always be expressed as

It is not the case that P.

there are usually better ways to express the statement $\neg P$.

1. P : The integer 77 is even.

   ____

   ____

   ____

2. P : $5^3 = 120$ $\neg P$:______________

3. P : The absolute value of the real number x is less than 5.

   ____

Compound Statements

Logical connectivity	write	read	meaning
Conjunction	$P \wedge Q$	P and Q	Both P and Q are true
Disjunction	$P \vee Q$	P or Q	P is true or Q is true

P: Ben is a student.

Q: Ben is a teaching assistant.

TRUTH TABLES

P	Q	$P \wedge Q$	$Q \wedge P$	$P \vee Q$	$Q \vee P$

EXAMPLE 5. *Rewrite the following open sentences (over* $\mathbf{R}$*) using disjunction or conjunction.*

(a) $P(x): \quad |x| \geq 10.$

(b) $P(x): \quad |x| < 10.$

(c) $P(x): \quad |4x + 7| \geq 23.$

Implications

DEFINITION 6. *Let* P *and* Q *be statements. The* **implication** $P \Rightarrow Q$ *(read "*P *implies* Q*") is the statement "If* P *is true, then* Q *is true."*

In implication $P \Rightarrow Q$, P is called *assumption, or hypothesis, or premise;* and Q is called *conclusion.*

The truth table for implication:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

EXAMPLE 7. P : *You pass the final exam.*
Q: *You pass the course.*

$P \Rightarrow Q$:

Alternative Expressions for $P \Rightarrow Q$**. Necessary and Sufficient Conditions**

If P, then Q. P implies Q. P only if Q. P is sufficient for Q.
Q if P. Q when P. Q is necessary for P.

In order for Q to be true it is sufficient that P be true.

or

Q must be true in order to P to be true.

REMARK 8. Note however, if $P \Rightarrow Q$ is true, then it is not necessary that P is true in order for Q to be true. Even if Q is true, P may be false.

Converse

DEFINITION 9. *The statement* $Q \Rightarrow P$ *is called a* **converse** *of the statement* $P \Rightarrow Q$.

EXAMPLE 10. *State the converse statement for implication in Example 7.*

EXAMPLE 11. P: *The function* $f(x) = \sin x$ *is differentiable everywhere.*
Q: *The function* $f(x) = \sin x$ *is continuous everywhere.*

$P \Rightarrow Q$ $Q \Rightarrow P$

Biconditional "⇔"

For statements P and Q,

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. The biconditional $P \Leftrightarrow Q$ is stated as

"*P is equivalent to Q.*" or "*P if and only if Q.*" (or "*P iff Q.*")

or as "*P is a necessary and sufficient condition for Q.*"

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

EXAMPLE 12. *Complete:*

(a) *The biconditional* "The number 17 is odd if and only if 57 is prime." *is* ____________.

(a) *The biconditional* "The number 24 is even if and only if 17 is prime." *is* ____________.

(a) *The biconditional* "The number 17 is even if and only if 24 is prime." *is* ____________.

Tautologies and Contradictions

Tautology: statement that is always true
Contradiction: statement that is always false

P	$\neg P$	$P \vee (\neg P)$	$P \wedge (\neg P)$
T			
F			

Methods to verify tautology/contradiction: truth table and deductive proof.

EXAMPLE 13. *Determine whether the following formula for the statements P and Q is a tautology, contradiction, or neither.*

$$\neg(P \Rightarrow Q) \Leftrightarrow P \wedge (\neg Q).$$

Logical Equivalence

DEFINITION 14. *Two compound statements are* **logically equivalent** *(write "$\equiv$") if they have the same truth tables, which means they both are true or both are false.*

Question: Are the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ logically equivalent? __________

EXAMPLE 15. *Let P and Q be statement forms. Determine whether the compound statements $\neg P \wedge Q$ and $\neg P \vee Q$ are logically equivalent (i.e. both true or both false).*

P	Q			

REMARK 16. Let P and Q be statements. The biconditional $P \Leftrightarrow Q$ is a tautology if and only if P and Q are logically equivalent.

THEOREM 17. *For statements P and Q,*

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q).$$

Some Fundamental Properties of Logical Equivalence

THEOREM 18. *For the statement forms P, Q and R,*

- $\neg(\neg P) \equiv$
- *Commutative Laws*
  $P \vee Q \equiv$
  $P \wedge Q \equiv$
- *Associative Laws*
  $P \vee (Q \vee R) \equiv$
  $P \wedge (Q \wedge R) \equiv$
- *Distributive Laws*
  $P \vee (Q \wedge R) \equiv$
  $P \wedge (Q \vee R) \equiv$
- *De Morgan's Laws*
  $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
  $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

Proof. Each part of the theorem is verified by means of a truth table.

P	Q					

Quantified Statements

EXAMPLE 19. *Consider the following* open sentence:

$$P(n): \ \frac{2n^2+5+(-1)^n}{2} \text{ is prime.}$$

How to convert this open sentence into a statement?

An open sentence can be made into a statement by using **quantifiers**.

Universal: $\forall x$ means for all/for every assigned value a of x.

Existential: $\exists x$ means that for some assigned values a of x.

Quantified statements

in symbols	in words
$\forall x \in D, P(x)$.	For every $x \in D$, $P(x)$. If $x \in D$, then $P(x)$.
$\exists x \in D \ni P(x)$	There exists x such that $P(x)$.

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. The variable that is not bound is called a **free** variable.

1. The area of a rectangle is its length times its width.
   Quantifiers:

2. A triangle may be equilateral.
   Quantifiers:

3. $15 - 5 = 10$
   Quantifiers:

4. A real-valued function that is continuous at 0 is not necessarily differentiable at 0.
   Quantifiers:

   ____

EXAMPLE 20. *For a triangle T, let*

$P(T)$: *T is equilateral* $Q(T)$: *T is isosceles.*

State $P(T) \Rightarrow Q(T)$ in a variety of ways:

EXAMPLE 21. *Consider the following open sentences*

$P(x)$: *x is a multiple of 4.* $Q(x)$: *x is even. Complete:*

- *"For every integer integer $x, P(x) \Rightarrow Q(x)$" is* ____________________.
- *$P(x)$ is a* ____________________ *condition for Q to be true.*
- *$Q(x)$ is a* ____________________ *condition for $P(x)$ to be true.*
- *$Q(x)$ is not a* ____________________ *condition for $P(x)$ to be true.*

EXAMPLE 22. *Consider the following open sentences*

$P(f)$: f is a differentiable function.

$Q(f)$: f is a continuous function.

Complete:

- *"For every real-valued function $f, P(f) \Rightarrow Q(f)$" is* ____________________.
- *"For every real-valued function $f, Q(f) \Rightarrow P(f)$" is* ____________________.
- *$Q(f)$ is a* ____________________ *condition for f to be differentiable, but not a* ____________________ *condition.*
- *$P(f)$ is a* ____________________ *condition for f to be continuous.*

EXAMPLE 23. *If m and n are odd integers then $m + n$ is even.*

Rewrite the statement in symbols. Then write its converse both in symbols and words.

EXAMPLE 24. *Rewrite the following statements in symbols using quantifiers. Introduce variables, where appropriate.*

a) *For every real number x, $x + 5 = 7$.*

____

b) *All positive real numbers have a square root.*

____

c) *The sum of an even integer and an odd integer is even.*

____

d) *For every integer n, either $n \leq 1$ or $n^2 \geq 4$.*

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NEGATIONS

DEFINITION 25. *If P is a statement, then the* **negation** *of P, written $\neg P$ (read "not P"), is the statement "P is false".*

1. All continuous functions are differentiable.

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2. P :There exist real numbers a and b such that $(a + b)^2 = a^2 + b^2$.

$\neg P$______________________________

Rules to negate statements with quantifiers:

$\neg(\forall x \in D, P(x)) \equiv$

$\neg(\exists x \in D \ni P(x)) \equiv$

$\neg(\forall x \in D, (P(x) \vee Q(x)) \equiv$

$\neg(\forall x \in D, (P(x) \wedge Q(x)) \equiv$

$\neg(\exists x \in D \ni (P(x) \vee Q(x)) \equiv$

$\neg(\exists x \in D \ni (P(x) \wedge Q(x)) \equiv$

EXAMPLE 26. *Negate the statements below using the following steps:*

1. *Rewrite P in symbols using quantifiers.*
2. *Express the negation of P in symbols using the above rules.*
3. *Express $\neg P$ in words.*

a) P : *If n is an odd integer then $3n+7$ is odd.*

P ____
$\neg P$ ____
$\neg P$ ____

b) P : *There exists a positive integer n such that $m(n+5) < 1$ for every integer m.*

P ____
$\neg P$ ____
$\neg P$ ____

c) P: *There exists a prime number p which is greater then 7 and less than 10.*

P ____
$\neg P$ ____
$\neg P$ ____

d) P : *For every even integer n there exists an integer m such that $n = 2m$.*

P____
$\neg P$____
$\neg P$____

e) P: *If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*

P____
$\neg P$____
$\neg P$____

Negating An Implication

THEOREM 27. *For statements P and Q,*

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q).$$

Proof.

REMARK 28. The negation of an implication is not an implication!

EXAMPLE 29. *Apply Theorem 27 to Negate the following statement:*[4]
S: If n is an integer and n^2 is a multiple of 4, then n is a multiple of 4.

S____
$\neg S$____

[4]Cf. Example 26(e)

EXAMPLE 30. *Express the following statements in the form "for all ... , if ... then ..." using symbols to represent variables. Then write their negations, again using symbols.*

(a) S : *Every octagon has eight sides.*

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(b) S : *Between any two real numbers there is a rational number.*

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