Mathematical Reasoning (Part II)

Integers and some of their basic properties and definitions

Let $a, b, c \in \mathbb{Z}$:

property	w.r.t.addition	w.r.t. multiplication
Closure	$a+b \in \mathbb{Z}$	$a \cdot b \in \mathbb{Z}$
Associative	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
Commutative	$a+b = b+a$	$ab = ba$
Distributive	$a(b+c) = ab+ac$	
Identity	$a+0=a$	$a \cdot 1 = a$ Note: $0 \neq 1$ and $a \cdot 0 = 0$.
Inverse	There exists a unique integer $-a = (-1) \cdot a$ such that $a+(-a)=0$	
Subtraction	$b-a := b+(-a)$	
No divisors of 0		If $ab=0$ then $a=0$ or $b=0$.
Cancellation	If $a+c=b+c$, then $a=b$.	If $ab=ac$ and $c \neq 0$, then $b=c$.

Order properties:

1. If $a<b$ and $b<c$ then $a<c$. (**transitivity**)
2. Exactly one of $a<b$ or $a=b$ or $a>b$ holds. (**trichotomy**)
3. If $a<b$, then $a+c<b+c$.
4. If $c>0$, then $a<b$ iff $ac<bc$.
5. If $c<0$, then $a<b$ iff $ac>bc$.

Mathematical definitions are always biconditional statements.

DEFINITION A. *An integer n is defined to be* **even** *if $n=2k$ for some integer k. An integer n is defined to be* **odd** *if $n=2k+1$ for some integer k.*

DEFINITION B. *The integers m and n are said to be* **of the same parity** *if m and n are both even, or both odd. The integers m and n are said to be* **of opposite parity** *if one of them is even and the other is odd.*

DEFINITION C. Let a and b be integers. We say that b **divides** a, written $b|a$, if there is an integer c such that $bc=a$. We say that b and c are **factors** of a, or that a is **divisible** by b and c.

FACT *Every integer is either even, or odd.*

DEFINITION 1. *A real number x is* **rational** *if $x = \frac{m}{n}$ for some integer numbers m and n. Also, x is* **irrational** *if it is not rational, that is*

Proving Statements Containing Implications

Most theorems (or results) are stated as implications.

Trivial and Vacuous Proofs[1]

Let $P(x)$ and $Q(x)$ be open sentences over a domain D. Consider the quantified statement $\forall x \in D, P(x) \Rightarrow Q(x)$, i.e.

For $x \in D$, if $P(x)$ then $Q(x)$. (#)

or *Let $x \in D$. If $P(x)$, then $Q(x)$.*

`The truth table for implication` $P(x) \Rightarrow Q(x)$ `for an arbitrary (but fixed) element` $x \in D$:

$P(x)$	$Q(x)$	$P(x) \Rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Trivial Proof If it can be shown that $Q(x)$ is true for all $x \in D$ (regardless the truth value of $P(x)$), then (#) is true (according the truth table for implications).

Vacuous Proof If it can be shown that $P(x)$ is false for all $x \in D$ (regardless of the truth value of $Q(x)$), then (#) is true (according the truth table for implications).

EXAMPLE 2. *Let $x \in \mathbb{R}$. If $x^6 - 3x^4 + x + 3 < 0$, then $x^4 + 1 > 0$.*

EXAMPLE 3. *Let $a, b \in \mathbb{R}$. If $a^2 + 2ab + b^2 + 1 \leq 0$, then $a^7 + b^7 \geq 7$.*

[1] These kind of proofs are rarely encountered in mathematics, however, we consider them as important reminders of implications.

DIRECT PROOFS

Let $P(x)$ and $Q(x)$ be open sentences over a domain D.

To prove (directly) a statement of the form "For all $x \in D$, $P(x)$ is true":

- Assume x is an arbitrary (but now fixed) element $x \in D$.
- Demonstrate that $P(x)$ is true.

EXAMPLE 4. *Let $n \in \mathbb{Z}$. Prove that if n is even, then $5n^5 + n + 6$ is even.*

To prove (directly) a statement of the form "For all $x \in D$, $P(x) \Rightarrow Q(x)$":

- Assume that $P(x)$ is true for an arbitrary (but now fixed) element $x \in D$.
- Draw out consequences of $P(x)$.
- Use these consequences to show that $Q(x)$ must be true as well for this element x.

REMARK 5. Note that if $P(x)$ is false for some $x \in D$, then $P(x) \Rightarrow Q(x)$ is ________ for this element x. This is why we need only be concerned with showing that $P(x) \Rightarrow Q(x)$ is true for all $x \in D$ for which $P(x)$ is true.

EXAMPLE 6. The following is an attempted proof of a result. What is the result and is the attempted proof correct?

Proof. Let a be an even integer and b be an odd integer. Then $a = 2n$ and $b = 2n + 1$ for some integer n. Therefore,

$$3a - 5b = 3(2n) - 5(2n + 1) = 6n - 10n - 5 = -4n - 5 = 2(-2n - 2) - 1.$$

Since $-2n - 2$ is an integer, $3a - 5b$ is odd. □

EXAMPLE 7. *Prove that the product of every two odd integers is odd.*

EXAMPLE 8. *Let x be an integer. Prove that if $5x - 7$ is odd, then $9x + 2$ is even.*

EXAMPLE 9. *Prove that the sum of every two rational numbers is rational.*

EXAMPLE 10. *Let* $a, b, c, d \in \mathbb{Z}$ *with* $a \neq 0$ *and* $b \neq 0$. *Prove the following:*

(a) *If* $a|b$ *and* $b|c$, *then* $a|c$.

(b) *If* $a|c$ *and* $b|d$, *then* $ab|cd$.

(c) *If* $a|c$ *and* $a|d$, *then for all* $x, y \in \mathbb{Z}$, $a|(cx + dy)$.

Contrapositive

DEFINITION 11. *The statement* $\neg Q \Rightarrow \neg P$ *is called the* **contrapositive** *of the statement* $P \Rightarrow Q$.

THEOREM 12. *For every two statements* P *and* Q*, the implication* $P \Rightarrow Q$ *and its contrapositive are logically equivalent; that is*

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$$

Proof.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

Conclusion To prove $P \Rightarrow Q$, we my choose instead to prove $\neg Q \Rightarrow \neg P$.

PROOF BY CONTRAPOSITIVE

Let $P(x)$ and $Q(x)$ be open sentences over a domain D. A proof by contrapositive of an implication is a direct proof of its contrapositive; that is **to prove that for all** $x \in D$**,** $P(x) \Rightarrow Q(x)$

- Assume that $\neg Q(x)$ is true for an arbitrary (but now fixed) element $x \in D$.
- Draw out consequences of $\neg Q(x)$.
- Use these consequences to show that $\neg P(x)$ must be true as well for this element x.
- It follows that $P(x) \Rightarrow Q(x)$ for all $x \in D$.

REMARK 13. If you use a contrapositive method, you must declare it in the beginning and then state **what is sufficient to prove.**

EXAMPLE 14. *Let* x *be an integer. If* $5x - 7$ *is even, then* x *is odd.*

THEOREM 15. *Let n be an integer. Then n is even if and only if n^2 is even.*

Proof.

REMARK 16. $(P \Leftrightarrow Q) \equiv (\neg P \Leftrightarrow \neg Q)$

COROLLARY 17. *Let n be an integer. Then n is odd iff n^2 is odd.*

EXAMPLE 18. *Let $x \in \mathbb{Z}$. Prove that if $2|(x^2-1)$ then $4|(x^2-1)$.*

EXAMPLE 19. *Let $x, y \in \mathbb{Z}$. If $7 \not| xy$, then $7 \not| x$ and $7 \not| y$.*

PROOF BY CASES

may be useful while attempting to give a proof of a statement concerning an element x in some set D. Namely, if x possesses one of two or more properties, then it may be convenient to divide a case into other cases, called *subcases.*

Result	Possible cases
$\forall n \in \mathbb{Z}, R(n)$	Case 1. $n \in \mathbb{E}$; Case 2. ______________
$\forall x \in \mathbb{R}, Q(x)$	Case 1. $x < 0$; Case 2. ______________ Case 3. $x > 0$
$\forall n \in \mathbb{Z}^+, P(n)$	Case 1. ______________; Case 2. $n \geq 2$.
$\forall x, y \in \mathbb{R} \ni xy \neq 0, P(x,y)$	Case 1. $xy < 0$; Case 2. ______________

EXAMPLE 20. *Prove that if n is an integer, then $n^2 + 3n + 4$ is an even integer.*

EXAMPLE 21. *Let $x, y \in \mathbb{Z}$. Prove that x and y are of opposite parity if and only if $x + y$ is odd.*

Disproving Statements

Case 1. Counterexamples

Let $S(x)$ be an open sentence over a domain D. If the quantified statement $\forall x \in D, S(x)$ is *false*, then its negation is true, i.e.

Such an element x is called a **counterexample** of the false statement $\forall x \in D, S(x)$.

EXAMPLE 22. *Disprove the statement:* *"If $n \in \mathbb{O}$, then $3|n^2+2$."*

Solution.

EXAMPLE 23. *Negate the statement: "For all $x \in D, P(x) \Rightarrow Q(x)$."*

The value assigned to the variable x that makes $P(x)$ true and $Q(x)$ false is a **counterexample** of the statement "For all $x \in D, P(x) \Rightarrow Q(x)$."

EXAMPLE 24. *S: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*
Question: Is the following "proof" valid?
Let $n = 6$. Then $n^2 = 6^2 = 36$ and 36 is a multiple of 4, but 6 is not a multiple of 4. Therefore, the statement S is FALSE.□

EXAMPLE 25. Disprove the following statement:
If a real-valued function is continuous at some point, then this function is differentiable there.

Case 2: Existence Statements

Consider the quantified statement $\exists x \in D \ni S(x)$. If this statement is *false*, then its negation is true, i.e.

EXAMPLE 26. *Disprove the statement:* "There exists an even integer n such that $3n + 5$ is even."

PROOF BY CONTRADICTION

To prove a statement S is true by contradiction:

- Assume that $\neg S$ is true.
- Deduce a contradiction.
- Then conclude that S is true.

REMARK 27. If you use a proof by contradiction to prove that S, you should alert the reader about that by saying (or writing) one of the following

- Suppose that the statement S is false.
- Assume, to the contrary, that the statement S is false.
- By contradiction, assume, that the statement S is false.

EXAMPLE 28. *Prove that there is no smallest positive real number.*

THEOREM 29. *Let S and C be statement forms. Then $\neg S \Rightarrow (C \wedge \neg C)$ is logically equivalent to S.*

Proof.

COROLLARY 30. *Let P, Q and C be statement forms. Then*

$$(P \Rightarrow Q) \equiv ((P \wedge \neg Q) \Rightarrow (C \wedge \neg C))$$

Proof.

To prove a statement $P \Rightarrow Q$ by contradiction:

- Assume that P is true.
- To derive a contradiction, assume that $\neg Q$ is true.
- Prove a false statement C, using negation $\neg(P \Rightarrow Q) \equiv (P \wedge \neg Q)$.
- Prove $\neg C$. It follows that Q is true. (The statement $C \wedge \neg C$ must be false, i.e. a contradiction.)

REMARK 31. If you use a proof by contradiction to prove that $P \Rightarrow Q$, your proof might begin with

- Assume, to the contrary, that the statement P is true and the statement Q is false.

  or
- By contradiction, assume, that the statement P is true and $\neg Q$ is true.

REMARK 32. If you use a proof by contradiction to prove the quantified statement

$$\forall x \in D, P(x) \Rightarrow Q(x),$$

then the proof begins by assuming the existence of a counterexample of this statement. Therefore, the proof might begin with:

- Assume, to the contrary, that there exists some element $x \in D$ for which $P(x)$ is true and $Q(x)$ is false.

  or
- By contradiction, assume, that there exists an element $x \in D$ such that $P(x)$ is true, but $\neg Q(x)$ is true.

EXAMPLE 33. *Prove that the sum of a rational and an irrational number is irrational.*

PROPOSITION 34. *If m and n are integers, then $m^2 \neq 4n+2$.*

Proof.

COROLLARY 35. *The equation $m^2 - 4n = 2$ has no integer solutions.*

COROLLARY 36. *If the square of an integer is divided by 4, the remainder cannot be equal 2.*

COROLLARY 37. *The square of an integer cannot be of the form $4n+2$, $n \in \mathbb{Z}$.*

PROPOSITION 38. *Let $a, b,$ and c be integers. If $a^2 + b^2 = c^2$ then a or b is an even integer.*

Proof.

Recall that a real number x is **rational** if $x = \frac{m}{n}$ for some integer numbers m and n. Note that if necessary, we may assume that the integers m and n have no common positive factors other than 1. (In other words, we may assume that every fraction can be *reduced to least terms.*)

PROPOSITION 39. *The number $\sqrt{2}$ is irrational.*

A Review of Three Proof Techniques

EXAMPLE 40. *Prove the following statement by a direct proof, by a proof by contrapositive and by a proof by contradiction:*

"If n is an even integer, then $5n + 9$ is odd."

How to prove (and not to prove) that $\forall x \in D, P(x) \Rightarrow Q(x)$.

	First step of "Proof"	**Technique**	**Goal**
1	Assume that there exists $x \in D$ such that $P(x)$ is true.		
2	Assume that there exists $x \in D$ such that $P(x)$ is false.		
3	Assume that there exists $x \in D$ such that $Q(x)$ is true.		
4	Assume that there exists $x \in D$ such that $Q(x)$ is false.		
5	Assume that there exists $x \in D$ such that $P(x)$ and $Q(x)$ are true.		
6	Assume that there exists $x \in D$ such that $P(x)$ is true and $Q(x)$ is false.		
7	Assume that there exists $x \in D$ such that $P(x)$ is false and $Q(x)$ is true.		
8	Assume that there exists $x \in D$ such that $P(x)$ and $Q(x)$ are false.		
9	Assume that there exists $x \in D$ such that $P(x) \Rightarrow Q(x)$ is true.		
10	Assume that there exists $x \in D$ such that $P(x) \Rightarrow Q(x)$ is false.		

Existence Proofs

An existence theorem can be expressed as a quantified statement
$\exists x \in D \ni S(x)$: There exists $x \in D$ such that $S(x)$ is true.

A proof of an existence theorem is called an existence proof.

EXAMPLE 41. *There exists real numbers a and b such that $\sqrt{a^2+b^2} = a+b$.*

Proof.

THEOREM 42. (**Intermediate Value Theorem of Calculus**) *If f is a function that is continuous on the closed interval $[a,b]$ and m is a number between $f(a)$ and $f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = m$.*

EXAMPLE 43. *Prove that following equation has a real number solution (a root) between $x = 2/3$ and $x = 1$:*

$$x^3 + x^2 - 1 = 0.$$

Uniqueness Proof

An element belonging to some prescribed set D and possessing a certain property P is **unique** if it is the only element of D having property P. Typical ways to prove uniqueness:

1. By a direct proof: *Assume that x and y are elements of D possessing property P and show that $x = y$.*

2. By a proof by contradiction: *Assume that x and y are distinct elements of D and show that $x = y$.*

EXAMPLE 44. *Prove that following equation has a unique real number solution (a root) between $x = 2/3$ and $x = 1$:*

$$x^3 + x^2 - 1 = 0.$$

Proof by Induction

"Domino Effect"

Step 1. The first domino falls.

Step 2. When any domino falls, the next domino falls.

Conclusion. All dominoes will fall!

THEOREM 45. *(*`First Principle of Mathematical Induction (PMI)`*) Let $P(n)$ be a statement about the positive integer n so that n is a free variable in $P(n)$.* ***Suppose the following:***

(PMI 1) *The statement $P(1)$ is true.*

(PMI 2) *For all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.*

Then *for all positive integers n, $P(n)$ is true.*

Strategy

The proof by induction consists of the following steps:

Basic Step: Verify that $P(1)$ is true.

Induction hypothesis: Assume that k is a positive integer for which $P(k)$ is true .

Inductive Step: With the assumption made, prove that $P(k+1)$ is true.

Conclusion: $P(n)$ is true for every positive integer n.

EXAMPLE 46. *Prove by induction the formula for the sum of the first n positive integers*

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}. \tag{1}$$

EXAMPLE 47. *Prove that* $3|8^n - 5^n$ *for every positive integer* n.

EXAMPLE 48. *Find the sum of all odd numbers from* 1 *to* $2n+1$ $(n \in \mathbb{Z}^+)$.