2. Sets

- **Set Terminology and Notation**

  Set is a well-defined collection of objects.

  Elements are objects or members of the set.

Describing a Set

- **Roster notation:**

  $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

- **Indicating a pattern:**

  $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

Set-builder notation:

The symbol "|" is read "such that".

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Set-builder notation (a more precise way of describing a set)

NOTATION 1. *Let $P(x)$ be an open sentence. Then the notation*

$$\{x|P(x)\} \quad or \quad \{x : P(x)\}$$

denotes the set of all elements x such that $P(x)$ is a true statement. (The symbol "|" is read "such that".)

When D is a set,

$$\{x \in D|\ P(x)\} = \{x|\ x \in D \quad \wedge \quad P(x)\}$$

EXAMPLE 2. *Use set-builder notation and to describe the following sets in two different ways:*

a) $\mathbf{O}$

b) $5\mathbf{Z}$

c) $\mathbf{N}$

d) $\mathbf{Q}$

EXAMPLE 3. *Prove or disprove: "If*If $A = \{x|x \in \mathbf{R} \wedge |x| = 1\}$, $B = \left\{x|x \in \mathbf{R} \wedge x^4 = 1\right\}$, and $C = \left\{x|x \in \mathbf{C} \wedge x^4 = 1\right\}$, then $A = B = C$."

Interval notation:

Intervals:

- bounded intervals:

1. closed interval $[a, b] =$
2. open interval $(a, b) =$
3. half-open,half-closed interval $(a, b] =$
4. half-closed,half-open interval $[a, b) =$

- unbounded intervals:

5. $[a, \infty) =$
6. $(a, \infty) =$
7. $(-\infty, a] =$
8. $(-\infty, a) =$
9. $(-\infty, \infty) =$

EXAMPLE 4. *Represent the following sets in interval notation when it is possible.*

a) $\{x \in \mathbf{R} | \ (x \geq 0) \wedge (x \in \mathbf{Z})\} =$

b) $\{x \in \mathbf{Z} | \ 3 \leq x < 10\} =$

c) $\{x \in \mathbf{R} | \ -2016 \leq x \leq 2017\} =$

Subsets

- Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!).
- If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$.
- If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

  Note that if $A = \{x \in D | \ P(x)\}$, then $A \subseteq D$.
- The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$.
- The **universal set** is the set that contains all of the elements for a problem, denoted by U.

EXAMPLE 5. *Let* $A, B \subseteq U$. *Then*

$A = B \Leftrightarrow \forall x \in U, (x \in A \Leftrightarrow x \in B)$

$A \subseteq B \Leftrightarrow \forall x \in U, (x \in A \Rightarrow x \in B)$

$A \subset B \Leftrightarrow$

Question: Let $A = \{n \in \mathbf{Z} | n \text{ is even}\}$, $B = \left\{n \in \mathbf{Z} | n^2 \text{ is even}\right\}$, and $C = \left\{n^2 | n \text{ is even}\right\}$. Are these sets the same?

EXAMPLE 6. *Let* $A = \{n \in \mathbb{Z} | n = 3t - 2 \text{ for some } t \in \mathbb{Z}\}$ *and* $B = \{n \in \mathbb{Z} | n = 3t + 1 \text{ for some } t \in \mathbb{Z}\}$. *Prove that* $A = B$.

Cardinality

infinite set

finite set

cardinality of A, $|A|$

EXAMPLE 7. *Let* A *and* B *be two sets.*

(a) ***TRUE/FALSE*** *If* $A = B$, *then* $|A| = |B|$.

(b) ***TRUE/FALSE*** *If* $|A| = |B|$, *then* $A = B$.

EXAMPLE 8. *Given* $A = \{0, 1, 2, \ldots, 8\}$, $B = \{1, 3, 5, 7\}$, $C = \{3, 5, 1, 7, 3, 1\}$, $D = \{5, 3, 1\}$, *and* $E = \emptyset$, *then which of the following are TRUE?*

(**a**)$B = C$ (**b**)$B \subseteq C$ (**c**)$B \subset C$ (**d**)$C \subseteq B$ (**e**)$D \subset B$

(**f**)$D \subseteq B$ (**g**)$B \subset D$ (**h**)$8 \in A$ (**i**) $\{4, 6\} \subset A$ (**j**)$1, 5 \subset A$

(**k**)$9 \notin C$ (**l**)$D \subseteq D$ (**m**)$\emptyset = 0$ (**n**)$0 \in E$ (**o**)$A \in A$

(**p**) $|A| = 8$ (**q**) $|C| = 7$ (**r**) $|E| = 0$ (**q**) $|B| = 5$

EXAMPLE 9. *Which of the following are TRUE?*

1. $\mathbf{Z}^+ \subset \mathbf{Z}$
2. $\mathbf{Z}^+ \subseteq \mathbf{Z}$
3. $\mathbf{N} \subseteq \mathbf{Z}^+$
4. $\mathbf{Z} \subset \mathbf{Q} \subseteq \mathbf{R}$

EXAMPLE 10. *Describe the set* $S = \{x \in \mathbf{R} |\ \sin x = 2\}$ *in another manner.*

Power set

EXAMPLE 11. *Give all the subsets of* $A = \{x, y\}$

DEFINITION 12. *Let* A *be a set. The power set of* A*, written* $P(A)$*, is*

$$P(A) = \{X |\ X \subseteq A\}.$$

EXAMPLE 13. *Find the following*

(a) $P\{x, y\}$

(b) $|P\{x, y\}|$

EXAMPLE 14. *Let* $A = \{-1, 0, 1\}$.

1. *Write all subsets of* A.

2. *Find all elements of power set of* A.

3. *Write 3 subsets of* $P(A)$.

4. *Find* $|P(A)|$

5. *Compute* $|P(P(A))|$

6. *What are* $|P(A)|$ *and* $|P(P(A))|$ *for an arbitrary set* A*?*

EXAMPLE 15. *Find*

(a) $P(\emptyset)$

(b) $P(P(\emptyset))$

(c) $P(\{-1\})$

(d) $P(\{\emptyset, \{\emptyset\}\})$

REMARK 16. Note that

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\}, \quad \emptyset \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \in \{\emptyset, \{\emptyset\}\},$$

as well as

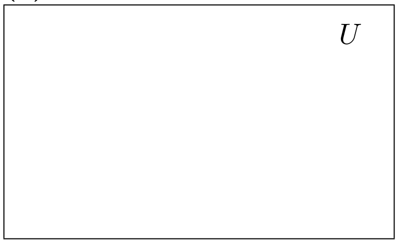
$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \notin \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \in P(\{\emptyset, \{\emptyset\}\}).$$

VENN DIAGRAMS

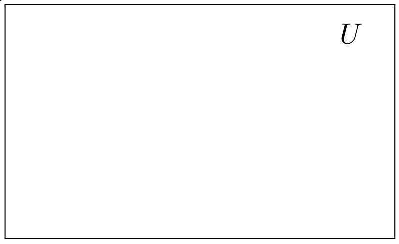
- a visual representation of sets (the universal set U is represented by a rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 17. *Use Venn diagrams to illustrate the following statements:*

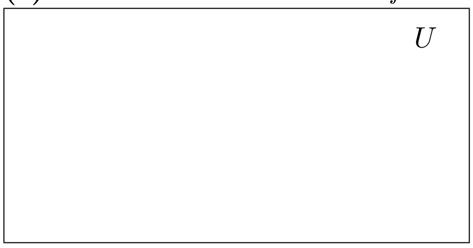
(a) $A = B$

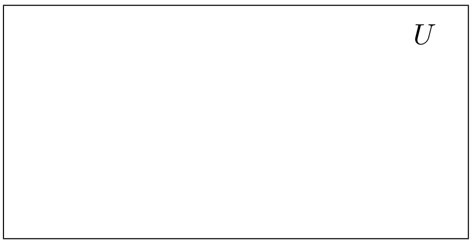


(b) $A \subset B \subset C$



(c) *A and B are not subsets of each other.*





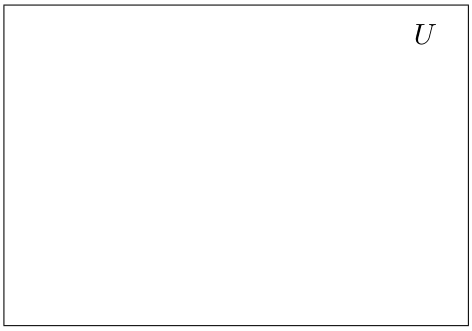
SET OPERATIONS

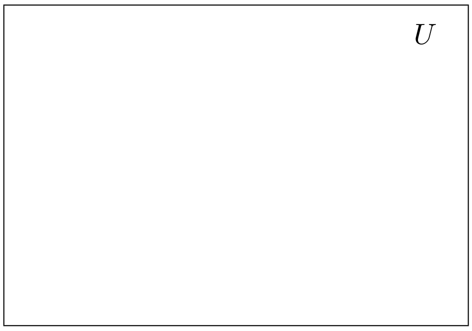
DEFINITION 18. *Let A and B be sets. The* **union** *of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:*

$$A \cup B = \{x | x \in A \vee x \in B\}.$$

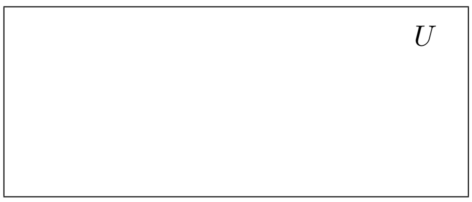
DEFINITION 19. *Let A and B be sets. The* ***intersection*** *of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:*

$$A \cap B = \{x | x \in A \ \wedge \ x \in B\}.$$

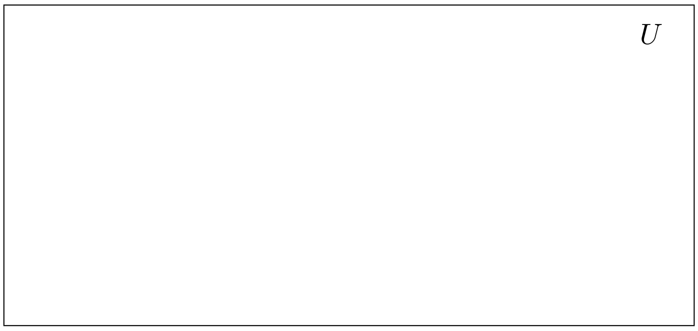




DEFINITION 20. *Let A and B be sets. The* **complement of** A **in** B *denoted $B - A$, is* $\{b \in B | b \notin A\}$.



REMARK 21. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.



EXAMPLE 22. *Let A be a subset of a universal set U. Prove the following*

(a) $\overline{\overline{A}} = A$.

(b) $\overline{\emptyset} = U$.

(c) $\overline{U} = \emptyset$

EXAMPLE 23. *Let $U = \{0, 1, 2, \ldots, 9, 10\}$ be a universal set, $A = \{0, 2, 4, 6, 8, 10\}$, and $B = \{1, 3, 5, 7, 9\}$. Find*

$$(\overline{A \cap B}) \cap (\overline{A \cup B}).$$

set notation	$=$	$\subset, \subseteq$	$\cup$	$\cap$	$\bar{\square}$	$\emptyset$
logical connectivity						

Cartesian Product

DEFINITION 24. *Let A and B be sets. The* **Cartesian product** *of A and B, written $A \times B$, is the following set:*

$$A \times B = \{(a, b) |\ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 25. *Given $A = \{0, 1\}$ and $B = \{4, 5, 6\}$.*

(a) *Does the pair $(6, 1)$ belong to $A \times B$?*

(b) *List the elements of $A \times B$.*

(c) *What is the cardinality of $A \times B$?*

(d) *List the elements of $A \times A \times A$.*

(e) *Does the triple $(1, 6, 4)$ belong to $A \times B \times B$?*

(f) *Describe the following sets $R \times R$, $R \times R \times R$.*

2. Sets (Part II: Proofs Involving Sets)

Fundamental properties of sets

THEOREM 26. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (`commutative`)
2. $A \cap B = B \cap A$ (`commutative`)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (`associative`)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (`associative`)
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (`distributive`)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (`distributive`)

   `DeMorgan's Laws:` *If A and B are the sets contained in some universal set U then*

7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
8. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow$
- $x \in A - B \Leftrightarrow$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- $x \in \bar{A} \Leftrightarrow x \notin A$

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

THEOREM 27. *The following statements are true for all sets A and B.*

1. $A \subseteq A \cup B$.

2. $A \cap B \subseteq A$.

3. *The empty set is a subset of every set. (Namely, for every set A, $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*

4. $A \cup \emptyset = A$.

5. $A \cap \emptyset = \emptyset$.

EXAMPLE 28. *Let A and B be sets. Show that $(A - B) \cap B = \emptyset$.*

PROPOSITION 29. *For every two sets A and B,*

$$A - B = A \cap \bar{B}$$

EXAMPLE 30. *Let A, B and C be sets. Prove that*

$$A - (B \cup C) = (A - B) \cup (A - C)$$

EXAMPLE 31. *For the sets A, B and C prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

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PROPOSITION 32. *Let $A, B,$ and C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.*

EXAMPLE 33. *Let* A, B, C *and* D *be sets. If* $A \subseteq C$ *and* $B \subseteq D$*, then* $A \times B \subseteq C \times D$*.*

EXAMPLE 34. *Prove the following statement. Let* A *and* B *be subsets of a universal set* U*. Then* $A \subseteq B \Leftrightarrow A \cup B = B$*.*

EXAMPLE 35. *Let* A *and* B *be subsets of a universal set* U*. Prove that*

$$A = A - B \Leftrightarrow A \cap B = \emptyset.$$