3 FUNCTIONS

3.1 Definition and Basic Properties

DEFINITION 1. *Let A and B be nonempty sets. A* **function** *f from A to B is a* `rule` *that assigns to each element in the set A one and only one element in the set B.*

We call A the **domain** *of f and B the* **codomain** *of f.*

We write $f : A \to B$ and for each $a \in A$ we write $f(a) = b$ if b is assigned to a.

Using diagram

EXAMPLE 2. *Let $A = \{2, 4, 6, 10\}$ and $B = \{0, 1, -1, 8\}$. Write out three functions with domain A and codomain B.*

Some common functions

- *Identity* function $i_A : A \to A$ maps every element to itself:
- *Linear* function $f : \mathbb{R} \to \mathbb{R}$ is defined by
- *Constant* function $f : \mathbb{R} \to \mathbb{R}$ is defined by

DEFINITION 3. *Two functions f and g are* **equal** *if they have the same domain and the same codomain and if $f(a) = g(a)$ for all a in domain.*

Image of a Function

EXAMPLE 4. *Discuss codomain of $f(x) = x^4$.*

DEFINITION 5. *Let $f : A \to B$ be a function. The* **image** *of f is*

$$\operatorname{Im}(f) = \{y \in B | y = f(x) \quad for \quad some \quad x \in A\}.$$

Note that $f(X)$ is the **image** *of the set X under f.*
The **graph** *of f is the set*

$$\{(a, b) \in A \times B | b = f(a)\}.$$

EXAMPLE 6. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \cos x$. Find $\operatorname{Im}(f)$ and $f([0, \pi/2])$.*

EXAMPLE 7. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |\cos x|$. Find $\operatorname{Im}(f)$.*

EXAMPLE 8. *$f : \mathbb{R} \to \mathbb{Z}$ is defined by $f(x) = [x]$, where $[x]$ means the greatest integer $\leq x$. Find $\operatorname{Im}(f)$.*

Rules

- For $f : A \to B$ to prove that $\text{Im}(f) = S$, use the following tautology:

$$(y \in S) \Leftrightarrow (\exists x \in A \ni f(x) = y).$$

- For $f : A \to B$ to prove that $f(X) = S$, $X \subset A$, use the following tautology

$$(y \in S) \Leftrightarrow (\exists x \in X \ni f(x) = y).$$

EXAMPLE 9. *Let* $S = \{y \in \mathbb{R} | y \geq 0\}$. *Prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = x^4$ *then* $\text{Im}(f) = S$.

EXAMPLE 10. *$f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=5x-4$. Find $f([0,1])$. Justify your answer.*

EXAMPLE 11. *$f:\mathbb{Z}\to\mathbb{Z}$ is defined by*

$$f(n)=\begin{cases} n-1 & if \ \ n \ \ is \ \ even, \\ n+1 & if \ \ n \ \ is \ \ odd. \end{cases}$$

Prove that $f(\mathbb{E})=\mathbb{O}$.

PROPOSITION 12. *Let A and B be nonempty sets and $f : A \to B$ be a function. If $X \subseteq Y \subseteq A$ then $f(X) \subseteq f(Y)$.*

Proof.

PROPOSITION 13. *Let A and B be nonempty sets and $f : A \to B$ be a function. If $X \subseteq A$ and $Y \subseteq A$ then*

(a) $f(X \cup Y) = f(X) \cup f(Y)$.

(b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

Proof

Inverse Image

DEFINITION 14. *Let $f : A \to B$ be a function and let W be a subset of its codomain (i.e. $W \subseteq B$). Then the* **inverse image** *of W (written $f^{-1}(W)$) is the set*

$$f^{-1}(W) = \{a \in A | f(a) \in W\}.$$

REMARK 15. This definition implies the following:

- $(x \in f^{-1}(W)) \Leftrightarrow (f(x) \in W)$
- If $W \subseteq \text{Im}(f)$ then $(S = f^{-1}(W)) \Rightarrow (f(S) = W)$

EXAMPLE 16. *Let $A = \{a, b, c, d, e, f\}$ and $B = \{7, 9, 11, 12, 13\}$ and let the function $g : A \to B$ be given by*

$$g(a) = 11, g(b) = 9, g(c) = 9, g(d) = 11, g(e) = 9, g(f) = 7.$$

Find

$f^{-1}(\{7, 9\}) =$

$f^{-1}(\{12, 13\}) =$

$f^{-1}(\{11, 12\}) =$

EXAMPLE 17. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 4$. Let $W = \{x \in \mathbb{R} | x > 0\}$. Find $f^{-1}(W)$.*

EXAMPLE 18. *$f : \mathbb{Z} \to \mathbb{Z}$ is defined by*

$$f(n) = \begin{cases} n/2 & if \;\; n \in \mathbb{E}, \\ n+1 & if \;\; n \in \mathbb{O}. \end{cases}$$

Compute

(a) $f^{-1}(\{6,7\}) =$

(b) $f^{-1}(\mathbb{O})$

PROPOSITION 19. *Let A and B be nonempty sets and $f : A \to B$ be a function. If W and V are subsets of B then*

(a) $f^{-1}(W \cup V) = f^{-1}(W) \cup f^{-1}(V)$.

(b) $f^{-1}(W \cap V) = f^{-1}(W) \cap f^{-1}(V)$.

Section 3.2 Surjective and Injective Functions

Surjective functions ("onto")

DEFINITION 20. *Let $f : A \to B$ be a function. Then f is* **surjective** *(or a surjection) if the image of f coincides with its codomain, i.e.*

$$\mathrm{Im} f = B.$$

Note: surjection is also called "onto".

`Proving surjection:`

We know that for all $f : A \to B$: ______________________

Thus, *to show that $f : A \to B$ is a surjection it is sufficient to prove that* ______________

In other words, *to prove that $f : A \to B$ is a surjective function it is sufficient to show that*

EXAMPLE 21. *Determine which of the following functions are surjective. Give a formal proof of your answer.*

(a) *Identity function*

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^5.$

(c) $f:\mathbb{Z}\to\mathbb{Z},\ f(n)=\begin{cases} n-2 & \text{if} \quad n\in\mathbb{E}, \\ 2n-1 & \text{if} \quad n\in\mathbb{O}. \end{cases}$

(d) $f:\mathbb{Z}\to\mathbb{Z},\ f(n)=\begin{cases} n+1 & \text{if} \quad n\in\mathbb{E}, \\ n-3 & \text{if} \quad n\in\mathbb{O}. \end{cases}$

Injective functions ("one to one")

DEFINITION 22. *Let $f : A \to B$ be a function. Then f is* **injective** *(or an injection) if whenever $a_1, a_2 \in A$ and $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$.*

Note: surjection is also called "onto". Using diagram:

EXAMPLE 23. *Given $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.*

(a) *Write out an injective function with domain A and codomain B. Justify your answer.*

(b) *Write out a non injective function with domain A and codomain B. Justify your answer.*

`Proving injection:`
Let $P(a_1, a_2) : a_1 \neq a_2$ and $Q(a_1, a_2) : \forall f(a_1) \neq f(a_2)$.

Then by definition f is injective if ____________________.

Using contrapositive, we have ____.

EXAMPLE 24. *Determine which of the following functions are injective. Give a formal proof of your answer.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[5]{x}$.

(b) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n/2 & \text{if} \quad n \in \mathbb{E}, \\ 2n & \text{if} \quad n \in \mathbb{O}. \end{cases}$

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x^5 + 5x^3 + 2x + 2014.$

Bijective functions

DEFINITION 25. *A function that is both surjective and injective is called* **bijective** *(or bijection.)*

EXAMPLE 26. *Determine which of the following functions are bijective. Give a formal proof of your answer.*

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3.$

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$

(c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b,$ *where* $a, b \in \mathbb{R}.$

EXAMPLE 27. *Show that* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is bijective.*

Permutations

DEFINITION 28. *Let* A *be any set. A bijection* $f : A \to A$ *is called a permutation of* A*.*

EXAMPLE 29. *Let* $A = \{1, 2, 3\}$ *and* $f : A \to A$ *is defined by* $f(1) = 3, f(2) = 1, f(3) = 2.$

EXAMPLE 30. *Identity function is a permutation.*

EXAMPLE 31. $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = x^3$.

REMARK 32. If $A = n$ then there exists $n!$ different permutations of A.

3.3 Composition and Invertible Functions

DEFINITION 33. *Let* A *and* B *be nonempty sets. We define*

$$F(A, B) =$$

the set of all functions from A *to* B.

If $A = B$, *we simply write* $F(A)$.

Composition of Functions

DEFINITION 34. *Let* A, B, *and* C *be nonempty sets, and let* $f \in F(A, B)$, $g \in F(B, C)$. *We define a function*

$$gf \in F(A, C),$$

called the **composition** *of* f *and* g, *by*

$$gf(a) =$$

EXAMPLE 35. *Let* $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{r, s, t, u, v\}$ *and define the functions* $f \in F(A, B)$, $g \in F(B, C)$ *by*

$$f = \{(1, b), (2, d), (3, a), (4, a)\}, \qquad g = \{(a, u), (b, r), (c, r), (d, s)\}.$$

Find gf. *What is about* fg*?*

EXAMPLE 36. *Let* $f, g \in \mathbf{R}$ *be defined by* $f(x) = e^x$ *and* $g(x) = x \sin x$. *Find* fg *and* gf.

EXAMPLE 37. *Let* $A = \mathbf{R} - \{0\}$ *and* $f \in F(A)$ *is defined by* $f(x) = 1 - \frac{1}{x}$ *for all* $x \in \mathbf{R}$. *Determine* fff.

EXAMPLE 38. *Let* $f, g \in F(\mathbf{Z})$ *be defined by*

$$f(n) = \begin{cases} n+4, & \text{if } n \in \mathbf{E} \\ 2n-3, & \text{if } n \in \mathbf{O} \end{cases} \qquad g(n) = \begin{cases} 2n-4, & \text{if } n \in \mathbf{E} \\ (n-1)/2, & \text{if } n \in \mathbf{O} \end{cases}$$

Find gf *and* fg.

PROPOSITION 39. *Let $f \in F(A, B)$ and $g \in F(B, C)$. Then*

i. *If f and g are surjections, then gf is also a surjection.*

`Proof.`

ii. *If f and g are injections, then gf is also an injection.*

`Proof.`

COROLLARY 40. *If f and g are bijections, then gf is also a bijection.*

PROPOSITION 41. *Let* $f \in F(A, B)$*. Then* $fi_A = f$ *and* $i_B f = f$*.*

Inverse Functions

DEFINITION 42. *Let* $f \in F(A, B)$*. Then* f *is* **invertible** *if there is a function* $f^{-1} \in F(B, A)$ *such that*

$$f^{-1}f = i_A \quad \text{and} \quad ff^{-1} = i_B.$$

If f^{-1} *exists then it is called the* **inverse** *function of* f*.*

REMARK 43. f is invertible if and only if f^{-1} is invertible.

PROPOSITION 44. *The inverse function is unique.*

`Proof.`

EXAMPLE 45. *The function* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is known to be bijective (*see Example 8, Section 3.2*). Determine the inverse* $f^{-1}(x)$*, where* $x \in \mathbb{R} - \{3\}$.

REMARK 46. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) =$ ________

The function $f(x) = 3x^5 + 5x^3 + 2x + 2014$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 47. *Let A and B be sets, and let $f \in F(A, B)$. Then f is invertible if and only if f is bijective.*