3 FUNCTIONS (part I)

3.1 Definition and Basic Properties

DEFINITION 1. *Let A and B be nonempty sets. A* **function** *f from the set A to the set B is a correspondence that assigns to each element a in the set A one and only one element b in the set B, which is denoted by $f(a)$.*

We call A the **domain** *of f and B the* **codomain** *of f.*

If $a \in A$ and $b \in B$ are such that $b = f(a)$, then b is called the **value** of f at a, or the **image** of a under f. We may also say that f **maps** a to b.

Using diagram

DEFINITION 2. *Two functions f and g are* **equal** *if they have the same domain and the same codomain and if $f(a) = g(a)$ for all a in domain.*

DEFINITION 3. *The* **graph** *of $f : A \to B$ is the set*

$$G_f = \{(a, b) \in A \times B | b = f(a)\}.$$

- We can determine a function from its domain, codomain, and graph.
- We can describe a function by formula, by listing its values, or by words.

functions

EXAMPLE 4. *Let $A = \{2, 4, 6\}$ and $B = \{a, b, c, d\}$. Determine in which of the following cases, f is function from A to B.*

(a) $f(2) = b, f(4) = a, f(6) = d$

(b) $f(2) = c, f(4) = c, f(6) = c$

(c) $f(2) = a, f(4) = b, f(6) = c, f(4) = d$

(d) $f(2) = c, f(6) = d$

Some common functions

- **Identity** function $i_A : A \to A$ maps every element to itself:

- **Polynomial** of degree n with real coefficients $a_0, a_1, \ldots, a_n$ is a function from $\mathbb{R}$ to $\mathbb{R}$

  Polynomials of degrees 0,1,2,3 are constant, linear, quadratic, cubic, respectively. by

DEFINITION 5. *Let A and B be nonempty sets. We define*

$$F(A, B) =$$

the set of all functions from A to B.
If $A = B$, we simply write $F(A)$.

EXAMPLE 6. *Let $f \in F(A, B)$ be defined by $f(x) = x^3 + 3$. In each of the following cases find its graph and illustrate it .*

(a) $A = B = \mathbb{R}$

(a) $A = \{-1, 0, 1\}$, $B = \mathbb{R}$

Image (or Range) of a Function

DEFINITION 7. *Let* $f \in F(A, B)$. *The* **image** *of* f *is*

$$\mathrm{Im}(f) = \{y \in B | y = f(x) \quad for \quad some \quad x \in A\}.$$

Using symbols complete the following

- $\mathrm{Im}(f) \subseteq$ __________
- $y \in \mathrm{Im}(f) \;\Leftrightarrow$ ______________________________
- $y \notin \mathrm{Im}(f) \;\Leftrightarrow$ ______________________________

EXAMPLE 8. $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = \cos x$. *Find* $\mathrm{Im}(f)$ *and* $f([0, \pi/2])$.

EXAMPLE 9. *Let* $f : [\frac{1}{3}, \infty) \to \mathbb{R}$ *be defined by* $f(x) = \sqrt{3x - 1}$ *and* $S = \{y \in \mathbb{R} |\ y \geq 0\}$. *Prove that* $Imf = S$.

Section 3.2 Surjective and Injective Functions

Surjective functions ("onto")

DEFINITION 10. *Let $f : A \to B$ be a function. Then f is* **surjective** *(or a surjection) if the image of f coincides with its codomain, i.e.*

$$\mathrm{Im} f = B.$$

Note: surjection is also called "onto".

`Proving surjection:`

We know that for all $f : A \to B$: _________________________

Thus, *to show that $f : A \to B$ is a surjection it is sufficient to prove that* _________________

In other words,
*to prove that $f : A \to B$ is a surjective function it is sufficient to show that*_________________

Question: How to disprove surjectivity?

EXAMPLE 11. *Let $f \in F(\mathbb{R})$ and $g \in F(\mathbb{R}, [0, \infty))$ defined by $f(x) = g(x) = x^4$. Determine whether the following are true*

(a) $\mathrm{Im}(f) = \mathrm{Im}(g)$

(b) $f = g$

(c) *f is surjective*

(d) *g is surjective*

EXAMPLE 12. *Prove that the function* $f \in F(\mathbb{R} - \{2\}, \mathbb{R} - \{3\})$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is surjective.*

Injective functions ("one to one")

DEFINITION 13. *Let* $f : A \to B$ *be a function. Then* f *is* **injective** *(or an injection) if whenever* $a_1, a_2 \in A$ *and* $a_1 \neq a_2$*, we have* $f(a_1) \neq f(a_2)$.

In other words, f *is injective if and only if the images of every two distinct points in the domain of* f *are distinct.*

EXAMPLE 14. *Given* $A = \{1, 2, 3\}$ *and* $B = \{3, 4, 5\}$*. Determine whether the following functions are injective.Justify your answer.*

(a) $f \in F(A, B)$ *defined by* $G_f = \{(1, 3), (2, 4), (3, 5)\}$

(b) $g \in F(A, B)$ *defined by* $G_g = \{(1, 5), (2, 4), (3, 4)\}$

`Proving injection:`
Let $P(a_1, a_2) : a_1 \neq a_2$ *and* $Q(a_1, a_2) : f(a_1) \neq f(a_2)$.

Then by definition f *is injective if* ____________________.

Using contrapositive, we have ______________________________________.
In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 15. *Prove or disprove injectivity of the following functions.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[5]{x}$.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = \begin{cases} n/2 & \text{if } n \in \mathbb{E}, \\ 2n & \text{if } n \in \mathbb{O}. \end{cases}$

`Discussion Exercise.`

- `Must a strictly increasing or decreasing function be injective?`

- `Must an injective function be strictly increasing or decreasing?`

EXAMPLE 16. *Prove or disprove injectivity of the following functions. In each case, $f \in F(\mathbb{R})$.*

(a) $f(x) = 3x^5 + 5x^3 + 2x + \pi$.

(b) $f(x) = x^{12} + x^8 - x^4 + 12$.

Bijective functions

DEFINITION 17. *A function that is both surjective and injective is called* **bijective** *(or bijection.)*

EXAMPLE 18. *Determine which of the following functions are bijective.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

3.3 Composition and Invertible Functions

Composition of Functions

DEFINITION 19. *Let A, B, and C be nonempty sets, and let $f \in F(A, B)$, $g \in F(B, C)$. We define a function*

$$g \circ f \in F(A, C),$$

called the **composition** *of f and g, by*

$$g \circ f(a) =$$

`Alternative notation for composition of` f `and` g`:` gf

`Using diagram`

EXAMPLE 20. *Let* $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{r, s, t, u, v\}$ *and define the functions* $f \in F(A, B)$, $g \in F(B, C)$ *by their graphs:*

$$G_f = \{(1, b), (2, d), (3, a), (4, a)\}, \qquad G_g = \{(a, u), (b, r), (c, r), (d, s)\}.$$

Find $g \circ f$. *What is about* $f \circ g$*?*

EXAMPLE 21. *Let* $f, g \in F(\mathbf{R})$ *be defined by* $f(x) = e^x$ *and* $g(x) = x \sin x$. *Find* $f \circ g$ *and* $g \circ f$.

PROPOSITION 22. *Let* $f \in F(A, B)$, $g \in F(B, C)$, *and* $h \in F(C, D)$. *Then*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

i.e. composition of functions is associative.

Proof.

EXAMPLE 23. *Let* $f \in F(\mathbf{R} - \{0, 1\})$ *be defined by* $f(x) = 1 - \frac{1}{x}$. *Determine* $f \circ f \circ f$.

PROPOSITION 24. *Let* $f \in F(A, B)$ *and* $g \in F(B, C)$. *Then*

i. *If* f *and* g *are surjections, then* gf *is also a surjection.*

`Proof.`

ii. *If* f *and* g *are injections, then* gf *is also an injection.*

`Proof.`

COROLLARY 25. *If* f *and* g *are bijections, then* gf *is also a bijection.*

PROPOSITION 26. *Let* $f \in F(A,B)$. *Then* $fi_A = f$ *and* $i_B f = f$.

Inverse Functions

DEFINITION 27. *Let* $f \in F(A,B)$. *Then* f *is* **invertible** *if there is a function* $f^{-1} \in F(B,A)$ *such that*

$$f^{-1}f = i_A \quad \text{and} \quad ff^{-1} = i_B.$$

If f^{-1} *exists then it is called the* **inverse** *function of* f.

Question: *What is the inverse of* f^{-1}*?*

REMARK 28. f is invertible if and only if f^{-1} is invertible.

EXAMPLE 29. *Discuss the inverse of a function defined by* $f(x) = \sin x$.

PROPOSITION 30. *The inverse function is unique.*

`Proof.`

EXAMPLE 31. *Show that the function* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is invertible and find its inverse function. (Note that the given function is bijective.)*

REMARK 32. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) =$ ________

The function $f(x) = 3x^5 + 5x^3 + 2x + 2016$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 33. *A function $f \in F(A, B)$ is invertible if and only if f is bijective.*

COROLLARY 34. *If a function $f \in F(A, B)$ is bijective, so is f^{-1}.*