# 3 FUNCTIONS (part II)

## Functions and Sets

### Image of a Set

DEFINITION 1. *Let $f : A \to B$ be a function. If $X \subseteq A$, we define $f(X)$, the* **image** *of $X$ under $f$, by*

$$f(X) = \{y \in B | y = f(x) \quad for \quad some \quad x \in X\}.$$

EXAMPLE 2. *Let $f \in F(A, B)$. Complete:*

**(a)** *If $X \subseteq A$ then $f(X) \subseteq$ ____ $\subseteq$ ____*

**(b)** $f(A) =$ ____

**(c)** $y \in f(X) \Leftrightarrow$ ______________________________

**(d)** $y \notin f(X) \Leftrightarrow$ ______________________________

EXAMPLE 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x$. Find the following*

**(a)** $f([-\pi/2, \pi/2])$

**(b)** $f([-\pi/2, 0])$

**(c)** $f([-\pi/2, \pi/2017])$

**(d)** $f([-\pi/4, \pi/4])$

**(e)** *Does $-\dfrac{1}{2017}$ belong to $f([-\pi/2, \pi/2])$?*

### Inverse Image

DEFINITION 4. *Let $f : A \to B$ be a function and let $W$ be a subset of its codomain (i.e. $W \subseteq B$). Then the* **inverse image** *of $W$ (written $f^{-1}(W)$) is the set*

$$f^{-1}(W) = \{x \in A | f(x) \in W\}.$$

The inverse image $f^{-1}(W)$ is a subset of its domain containing all preimages of points from $W$ under $f$.

EXAMPLE 5. *Let $f \in F(A, B)$ and $W \subseteq B$. Complete:*

**(a)** $f^{-1}(W) \subseteq$ ____ $\subseteq$ ____

**(b)** $x \in f^{-1}(W) \Leftrightarrow$ ____________________________

**(c)** $x \notin f^{-1}(W) \Leftrightarrow$ ____________________________

**(d)** *If $x \in A$ and $y \in B$, then*

$$f(x)\_\_\_\_B, \quad f(\{x\})\_\_\_\_B, \quad , f^{-1}(y)\_\_\_\_A, \quad f^{-1}(\{y\})\_\_\_\_A,$$

EXAMPLE 6. *Let $f \in F(\mathbb{R})$ be defined by $f(x) = x^4$. Find the following:*

**(a)** $f([1, 2]) =$

**(b)** $f([-2, -1]) =$

**(c)** $f^{-1}([1, 16]) =$

**(d)** $f^{-1}([-16, -1]) =$

**(e)** $f^{-1}([-1, 1]) =$

**(f)** $f((-\infty, \infty)) =$

**(g)** $f^{-1}(\mathbb{R}) =$

**(h)** $f^{-1}(\mathbb{R}^+) =$

**(i)** $f(f^{-1}([-16, -1])) =$

**(j)** $f(f^{-1}([-1, 1])) =$

**(k)** *Let $X, Y \subseteq \text{dom}(f)$ and $W \subset \text{codom}(f)$. Determine the truth or falsehood of the following*

1. $f^{-1}(f(X)) = X$

2. $f(f^{-1}(W)) = W$

3. $f(X) \cap f(Y) = f(X \cap Y)$

**Summary**

Let $f : A \to B$. Then above definitions imply the following tautologies

- $(y \in \mathrm{Im}(f)) \Leftrightarrow (\exists x \in A \ni f(x) = y)$.
- $(y \in f(X)) \Leftrightarrow (\exists x \in X \ni f(x) = y)$.
- $(x \in f^{-1}(W)) \Leftrightarrow (f(x) \in W)$.

Also note that

- If $W \subseteq \mathrm{Im}(f)$ then $(S = f^{-1}(W)) \Rightarrow (f(S) = W)$.

EXAMPLE 7. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x - 4$. Find $f([0, 1])$. Justify your answer.*

EXAMPLE 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 4$ and let $W = \{x \in \mathbb{R} | x > 0\}$. Find $f^{-1}(W)$.*

PROPOSITION 9. *Let $f \in F(A, B)$. If $X \subseteq Y \subseteq A$ then $f(X) \subseteq f(Y)$.*

*Proof.*

PROPOSITION 10. *Let $f \in F(A, B)$. If $W$ and $V$ are subsets of $B$ then*

$$f^{-1}(W \cup V) = f^{-1}(W) \cup f^{-1}(V).$$

*Proof.*

EXAMPLE 11. *Let $A$ and $B$ be sets and $X$ and $Y$ be subsets of $A$. Let $f \in F(A, B)$.*

**(a)** *Prove that $f(X \cap Y) \subseteq f(X) \cap f(Y)$.*

**(b)** *Give an example of a function $f \in F(A, B)$ for some $A$ and $B$ for which $f(X \cap Y) \neq f(X) \cap f(Y)$ for some subsets $X$ and $Y$ of $A$.*

**(c)** *Prove that if, in addition, $f$ is an injective function, then $f(X \cap Y) = f(X) \cap f(Y)$.*

EXAMPLE 12. *Let $A$ and $B$ be sets and $W$ be a subset of $B$. Let $f \in F(A, B)$.*

**(a)** *Prove that $f(f^{-1}(W)) \subseteq W$.*

**(b)** *Give an example of a function $f \in F(A, B)$ for some $A$ and $B$ and subset $W$ of $B$ such that $f(f^{-1}(W)) \neq W$.*

**(c)** *Prove that if, in addition, $f$ is an surjective function, then $f(f^{-1}(W)) = W$.*