5.5: Congruences

Congruences and their properties

Discuss the following problem:

EXAMPLE 1. *Are there any integers x and y such that $x^2 = 4y + 3$?*

Recall that congruence mod n is an equivalence relation on $\mathbf{Z}$, i.e.

1.

2.

3.

PROPOSITION 2. *Let $a, b, c, d \in \mathbf{Z}$ and let $n \in \mathbf{Z}^+$. Then*

1. *If $a \equiv b (\text{mod } n)$ and $c \equiv d (\text{mod } n)$ then $a + c \equiv b + d (\text{mod } n)$.*
2. *If $a \equiv b (\text{mod } n)$ and $c \equiv d (\text{mod } n)$ then $ac \equiv bd (\text{mod } n)$.*
3. *If $ab \equiv ac (\text{mod } n)$ and $\gcd(a, n) = 1$ then $b \equiv c (\text{mod } n)$.*

Proof.

REMARK 3. If $\gcd(a,n) \neq 1$, then (3) maybe false.

COROLLARY 4. *If* $a \equiv b \pmod{n}$ *then* $a^k \equiv b^k \pmod{n}$ *for every* $k \in \mathbf{Z}^+$.

EXAMPLE 5. *Prove that* $7|6^{1000} - 1$ *and* $7|6^{1001} + 1$.

EXAMPLE 6. *What is the last digit of* 7^{1258}?

PROPOSITION 7. *Let $n \in \mathbf{Z}$, $n > 1$. If $a \in \mathbf{Z}$, then a is congruent modulo n to exactly one of the integers* $0, 1, 2, \ldots, n-1$.

EXAMPLE 8. *Show that square of any integer is congruent to* 0 *or to* 1 *(modulo 4). Then derive another proof of Example 1.*

Recall the definition of **congruence class** of a modulo n:

$$[a] = \{x \in \mathbf{Z} |\ x \equiv a (\text{mod } n)\}.$$

Note that

1. For any integer a, $[a]$ is a set, not an integer.
2. If $0 \le a < n$, then $[a]$ can be described as the set of integers that give a remainder of a when divided by n. (In this case we call a a standard representative of $[a]$.)
3. If $[a] = [b]$, it does not mean $a = b$, only that $a \equiv b(\text{mod } n)$ or that a and b give the same reminder when divided by n.

The set of congruence classes. Modular Arithmetic

Consider the following partition of $\mathbf{Z}$ by set of congruence classes:

$$\mathbf{Z}_n = \{[0], [1], [2], \dots, [n-1]\}$$

Addition on $\mathbf{Z}_n$: $[a] + [b] = [a + b]$

Multiplication on $\mathbf{Z}_n$: $[a][b] = [ab]$

EXAMPLE 9. *Give addition and multiplication tables for $\mathbf{Z}_n$ for $n = 2, 3$.*

$n = 2$

$+$	$[\mathbf{0}]$	$[\mathbf{1}]$
$[\mathbf{0}]$		
$[\mathbf{1}]$		

$\cdot$	$[\mathbf{0}]$	$[\mathbf{1}]$
$[\mathbf{0}]$		
$[\mathbf{1}]$		

$n = 3$

$+$	$[\mathbf{0}]$	$[\mathbf{1}]$	$[\mathbf{2}]$
$[\mathbf{0}]$			
$[\mathbf{1}]$			
$[\mathbf{2}]$			

$\cdot$	$[\mathbf{0}]$	$[\mathbf{1}]$	$[\mathbf{2}]$
$[\mathbf{0}]$			
$[\mathbf{1}]$			
$[\mathbf{2}]$			

EXAMPLE 10. *Compute*

(a) *in* $\mathbf{Z}_6$,

$[2][3] =$

$[2][4] =$

(b) *in* $\mathbf{Z}_{11}$,

$[6][7] =$

$[25] + [22] =$

THEOREM 11. *Let $n \in \mathbf{Z}$, $n > 1$.*

1. *Addition in $\mathbf{Z}_n$ is commutative*

2. *Addition in $\mathbf{Z}_n$ is associative*

3. *$[0]$ is the identity of $\mathbf{Z}_n$ w.r.t. addition:*

4. *Every element of $\mathbf{Z}_n$ has an inverse w.r.t. addition. Namely, for every $a \in \mathbf{Z}$ the additive inverse of $[a]$ is $[-a]$.*

THEOREM 12. *Let* $n \in \mathbf{Z}$, $n > 1$.

1. *Multiplication in* $\mathbf{Z}_n$ *is commutative*

2. *Multiplication in* $\mathbf{Z}_n$ *is associative*

3. $[1]$ *is the multiplicative identity of* $\mathbf{Z}_n$*:*

4. *The following distributive laws hold:*

$$[a]([b] + [c]) =$$

$$([a] + [b])[c] =$$

DEFINITION 13. *An element* $[a] \in \mathbf{Z}_n$ *has an inverse w.r.t. multiplication if there exists* $[x] \in \mathbf{Z}_n$ *such that* $[a][x] = [1]$.

EXAMPLE 14. $[5]$ *is invertible in* $\mathbf{Z}_9$ *because*

However, $[3]$ *and* $[6]$ *are not invertible in* $\mathbf{Z}_9$ *because*

THEOREM 15. *Let* $[a] \in \mathbf{Z}_n$. *Then* $[a]$ *has a multiplicative inverse if and only if* a *and* n *are relatively prime, i.e.* $\gcd(a, n) = 1$.

Proof.

EXAMPLE 16. **(a)** *Is* [51] *invertible in* $\mathbf{Z}_{65}$ *w.r.t. multiplication? If yes, find its inverse.*

(b) *Find the least positive integer* x *that satisfies the following congruence*

$$51x \equiv 3 (\text{mod } 65)$$

Fermat's Little Theorem

Question: Which of the following are true for any integer a?

- $2 | a^2 - a$
- $3 | a^3 - a$
- $4 | a^4 - a$
- $5 | a^5 - a$

What is wrong with number 4?

Fermat's Little Theorem. *For any prime integer p and $a \in \mathbf{Z}$,*

$$p | a^p - a.$$

Equivalently

Alternative version of Fermat's Little Theorem. For any prime integer p and $a \in \mathbf{Z}$ such that a and p are relatively prime, i.e. $\gcd(a, p) = 1$, one has

$$p | a^{p-1} - 1 \quad \text{or} \quad a^{p-1} \equiv 1 (\text{mod } p).$$

EXAMPLE 17. *Find the remainder if 7^{985} is divided by 13.*