

12 Vectors and Geometry of Space

12.1: Three-dimensional Coordinate System

The three-dimensional coordinate system consists of the **origin** O and the **coordinate axes**: x -axis, y -axis, z -axis. The coordinate axes determine **3 coordinate planes**: the xy -plane, the xz -plane and yz -plane. The coordinate planes divide space into 8 parts, called octants.

Representation of point $P(a, b, c)$ and its projections on the coordinate planes:

EXAMPLE 1. Describe in words the regions of \mathbb{R}^3 represented by the following equation:

(a) $z = 0$

(b) $y = 0$

(c) $x = 0$

Note that in \mathbb{R}^2 the graph of the equation involving x and y is a curve. In \mathbb{R}^3 an equation in x, y, z represents a **surface**. (It does not mean that we can't graph curves in \mathbb{R}^3 .)

EXAMPLE 2. Sketch the graph of $x^2 + y^2 - 1 = 0$ in $\mathbb{R}^2, \mathbb{R}^3$.

A **cylinder** is a surface of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

An equation that contains only two of the variables x, y, z represents a **cylindrical surface** in \mathbb{R}^3 .
How to graph cylindrical surface:

1. graph the equation in the coordinate plane of the two variables that appear in the given equation;
2. translate that graph parallel to the axis of the missing variable.

EXAMPLE 3. Sketch the graph of $(x + 2)^2 + (y - 4)^2 = 1$ in \mathbb{R}^3

EXAMPLE 4. Sketch the graph of $y = x^2$ in \mathbb{R}^3

EXAMPLE 5. Let S be the graph of $x^2 + z^2 - 10z + 21 = 0$ in \mathbb{R}^3 .

(a) Describe S .

(b) The intersection of S with the xz - plane is _____

(c) The intersection of S with the yz - plane is _____

(d) The intersection of S with the xy - plane is _____

Spheres

- **Distance formula in \mathbb{R}^3 :** The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXAMPLE 6. Show that the equation $x^2 + y^2 + z^2 + 2x - 4y + 8z + 17 = 0$ represents a sphere, and find its center and radius.

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k$$

- If $k > 0$ then the graph of this equation is _____
- If $k = 0$, then the graph is _____
- If $k < 0$ then _____

Regions in \mathbb{R}^3

EXAMPLE 7. Describe the set of all points in \mathbb{R}^3 whose coordinates satisfy the following inequality:
 $x^2 + y^2 < 16$

EXAMPLE 8. Describe the following region: $\{(x, y, z) | 9 \leq x^2 + y^2 + z^2 \leq 16\}$