## 12.2 & 12.3: Vectors and the Dot Product

DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point $P$.

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point $A(1, 2, 3)$ and terminal point $B(3, 2, -1)$. What is the position vector of the point $A$?*

**Vector Arithmetic:** Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

  TRIANGLE LAW PARALLELOGRAM LAW

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow$$

**The magnitude or length** of $a = \langle a_1, a_2, a_3\rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0,0,0\rangle$, $|0| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector in the same direction as a:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ *The process of multiplying a vectoe* **a** *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector **a** can be determined by its length and the angle from the positive $x$-axis:

In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

$$\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},$$

i.e. **a** is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2,5,0)$ *and* $N(0,0,4)$.

**Standard Basis Vectors**:
$\mathbf{i} = \langle 1,0,0\rangle$
$\mathbf{j} = \langle 0,1,0\rangle$
$\mathbf{k} = \langle 0,0,1\rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
We have:

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle =$$

EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3\rangle$ *and* $\mathbf{b} = \langle 3, 1, 2\rangle$. *Find*

**(a)** $|\mathbf{b} - \mathbf{a}|$.

**(b)** *a unit vector that has the same direction as* $\mathbf{b}$.

**Dot Product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta,$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

<u>Component Formula</u> for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If $\theta$ is the *angle* between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \underline{\hspace{6cm}}$$

DEFINITION 5. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

EXAMPLE 6. *For two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *prove that*

**(a)**

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

**(b)**

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

EXAMPLE 7. *For what value(s) of $c$ are the vectors $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ orthogonal?*

EXAMPLE 8. *The points $A(6, -1, 0)$, $B(-3, 1, 2)$, $C(2, 4, 5)$ form a triangle. Find angle at $A$.*