12.5: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle$.

Vector equation	Parametric equations	Symmetric equations
$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$, $-\infty < t < \infty$	$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, $-\infty < t < \infty$	If $abc \neq 0$ then $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ If, for example, $a = 0$ then the symmetric equations have the form: $x = x_0$, $\frac{y-y_0}{b} = \frac{z-z_0}{c}$

EXAMPLE 1. *Complete the following.*

(a) *The equation* $\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle 4, 5, 6\rangle$ *is a* ______________ *equation of the line passing through the point* ______________ *and parallel to the vector* $\mathbf{v} =$ ______________.

(b) *The equation* $\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\mathbf{j}$ *is a* ______________ *equation of the line passing through the point* ______________ *and parallel to the* ___*-axis.*

(c) *The equations* $x = 2 - t, y = -t, z = 5$ *are* ______________ *equations of the line passing through the point* ______________ *and parallel to the vector* $\mathbf{v} =$ ______________.

(d) *The equations* $\frac{x-4}{5} = y + 1 = \frac{z}{-3}$ *are* ______________ *equations of the line passing through the point* ______________ *and parallel to the vector* $\mathbf{v} =$ ______________.

(e) *The equations* $\frac{x-4}{5} = y + 1, z = 2$ *are* ______________ *equations of the line passing through the point* ______________ *and parallel to the vector* $\mathbf{v} =$ ______________.

EXAMPLE 2. *Find vector equation of the line passing through the point* $(3,-4,1)$ *and parallel to the vector* $\mathbf{v} = \langle 7, 0, -1\rangle$

Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*
2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 3. *Consider the line* L *that passes through the points* $A(1,1,1)$ *and* $B(2,3,-2)$.

(a) *Find parametric equations of* L.

(b) *Find point* C *at that the* L *intersects the* yz*-plane.*

(c) *Find parametric equations describing the line segment joining the points* A *and* C.

EXAMPLE 4. *Determine whether the lines*

$$L_1: \quad x-1=\frac{y+2}{3}=\frac{z-4}{-1}$$

and

$$L_2: \quad x=2t, \quad y=3+t, \quad z=-3+4t$$

are parallel, skew, or intersecting.

Planes

Planes parallel to the coordinate planes:

Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.

Vector equation of the plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$.

Scalar equation of plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 5. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

EXAMPLE 6. *Determine the equation of the plane through the points* $A(1, 1, 1)$, $B(0, 1, 0)$ *and* $C(1, 2, 3)$.

- Two planes are **parallel** if their normal vectors are parallel.
- Two planes are **orthogonal** if their normal vectors are orthogonal.
- If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 7. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

(a) *Find normal vectors corresponding to these planes.*

$$\vec{n}_1 =$$

$$\vec{n}_2 =$$

$$\vec{n}_3 =$$

$$\vec{n}_4 =$$

(b) *Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.*

(i) P_1 *and* P_2

(ii) P_1 *and* P_3

(iii) P_1 *and* P_4

Line as an intersection of two non parallel planes:

$$L : \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

The direction vector of L is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2$.

EXAMPLE 8. *Find an equation of the line given as intersection of two planes:*

$$\begin{aligned} x - y + 3z &= 0 \\ x + y + 4z &= 2 \end{aligned}$$