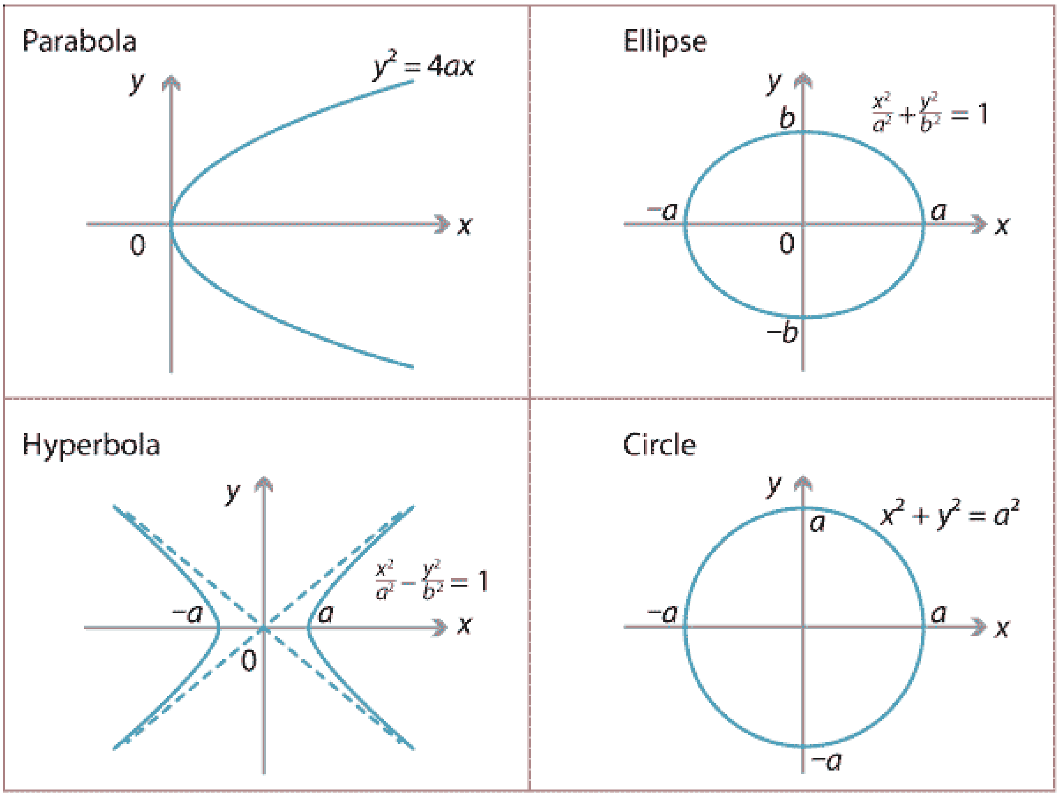
12.6: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.



The most general second-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0, \tag{1}$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

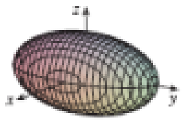
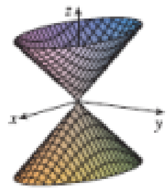
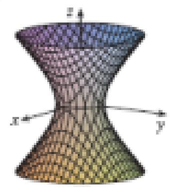
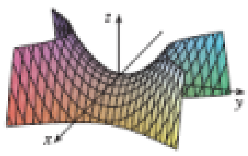
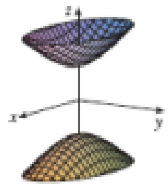
By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories:

ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders. (shown in the table below.)

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z=k$), yz-traces (by $x=0$) and xz-traces (by $y=0$).
3. Intercepts (in some cases).

To find the equation of a trace substitute the equation of the plane into the equation of the surface.

Note, in the examples below *the constants* $a, b,$ *and* c *are assumed to be positive.*

EXAMPLE 1. *Use traces to sketch the following quadric surfaces:*

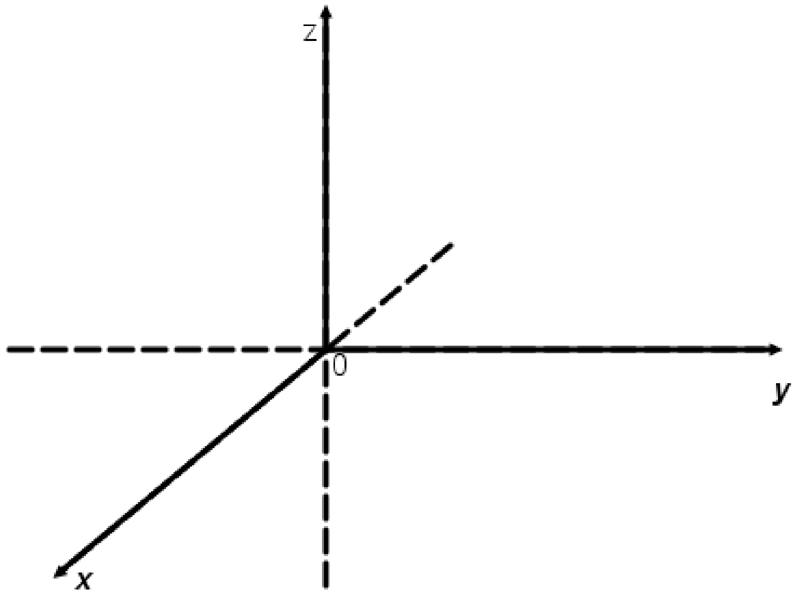
(a)

$$x^2+\frac{y^2}{16}+\frac{z^2}{9}=1$$

Solution

- *Find* **intercepts***:*
  - x*-intercepts: if* $y=z=0$ *then* $x=$
  - y*-intercepts: if* $x=z=0$ *then* $y=$
  - z*-intercepts: if* $x=y=0$ *then* $z=$

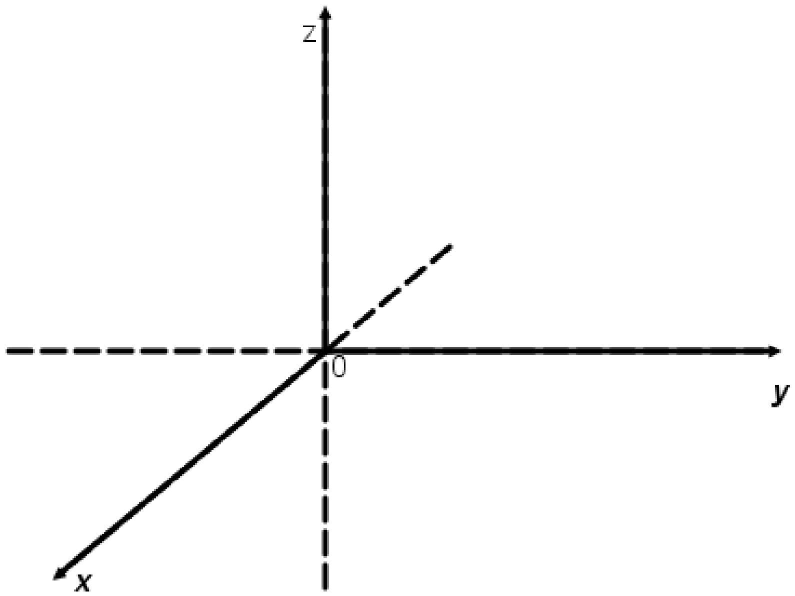
- *Obtain* **traces of**:
  - *the xy-plane: plug in* $z = 0$ *and get* $x^2 + \frac{y^2}{16} = 1$
  - *the yz-plane: plug in* $x = 0$ *and get*
  - *the xz-plane: plug in* $y = 0$ *and get*
  - *plug in* $z = k$
  - *plug in* $x = k$
  - *plug in* $y = k$



(b)

$$z^2 = x^2 + \frac{y^2}{9}$$

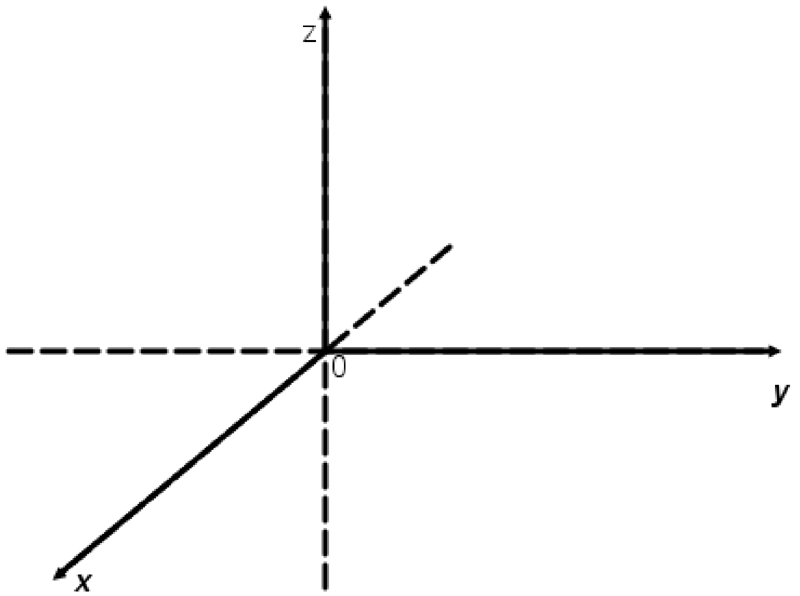
Plane	*Trace*
$z = k$	
$x = 0$	
$y = 0$	



(c)

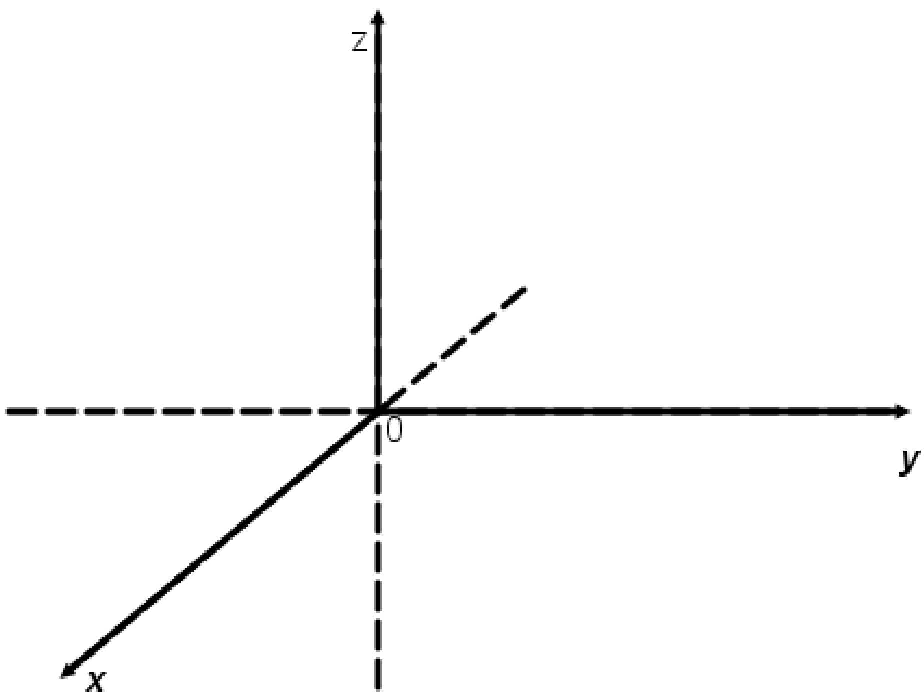
$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

Plane	*Trace*
$z = k$	
$x = 0$	
$y = 0$	



TRANSLATIONS AND REFLECTIONS OF QUADRIC SURFACES

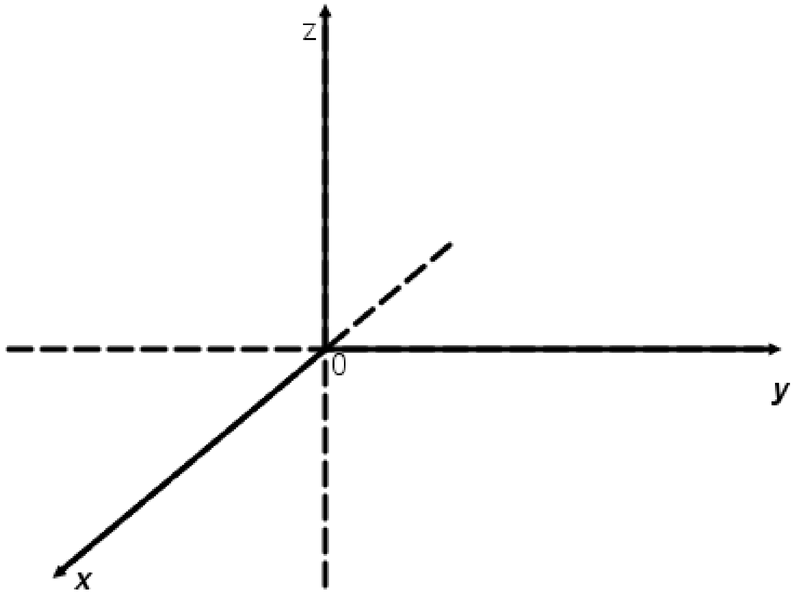
EXAMPLE 2. *Describe and sketch the surface* $z = (x+4)^2 + (y-2)^2 + 5$.



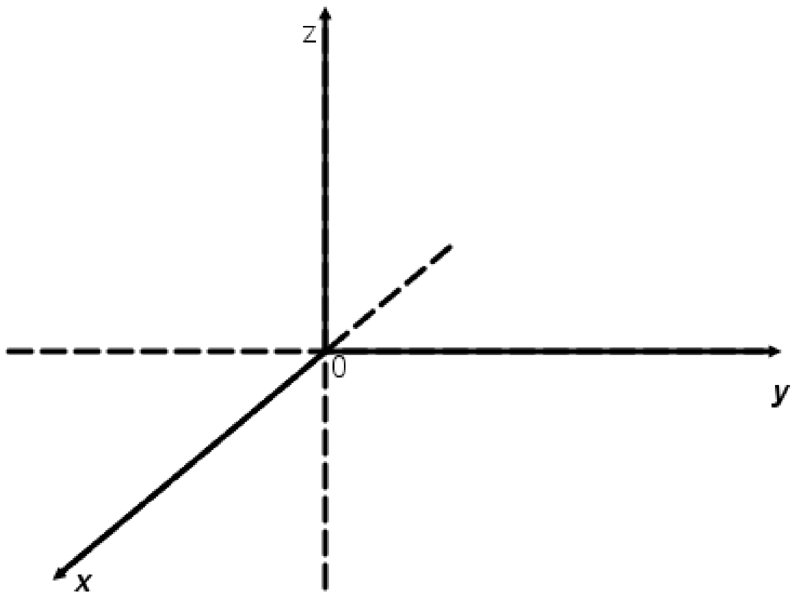
Note that **replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.**

EXAMPLE 3. *Identify and sketch the surface.*

(a) $z = -(x^2 + y^2)$



(b) $y^2 = x^2 + z^2$



EXAMPLE 4. *Classify and sketch the surface*

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$