

## 13.1: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let  $\mathbf{r}(t)$  be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where  $x(t), y(t), z(t)$  are functions of one variable and they are called the **component functions**.

A vector function  $\mathbf{r}(t)$  is *continuous* if and only if its component functions  $x(t), y(t), z(t)$  are continuous.

*Space curve is given by parametric equations:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), t \text{ in } I\},$$

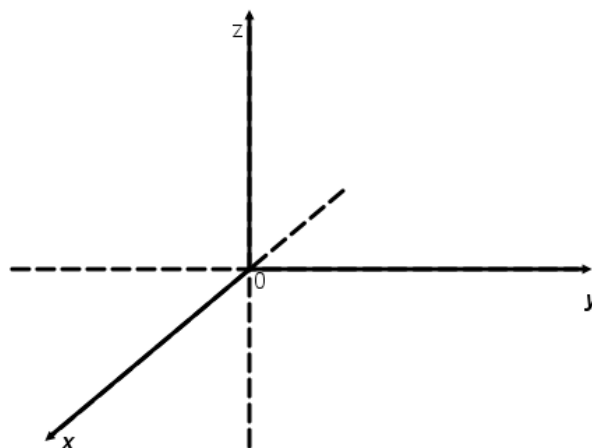
where  $I$  is an interval and  $t$  is a **parameter**.

*FACT: Any continuous vector-function  $\mathbf{r}(t)$  defines a space curve  $C$  that is traced out by the tip of the moving vector  $\mathbf{r}(t)$ .*

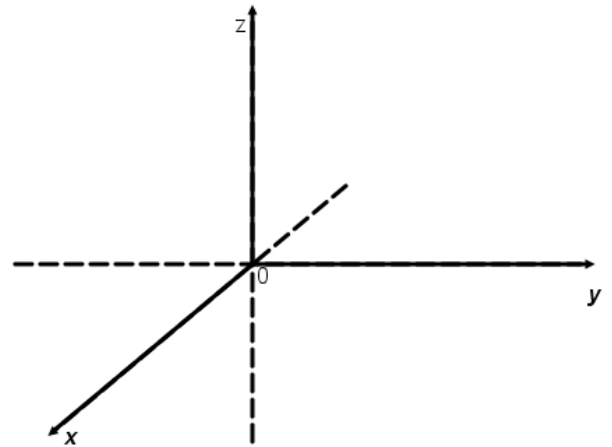
*Any parametric curve has a **direction of motion** given by increasing of parameter.*

EXAMPLE 1. Describe the curve defined by the vector function (indicate direction of motion):

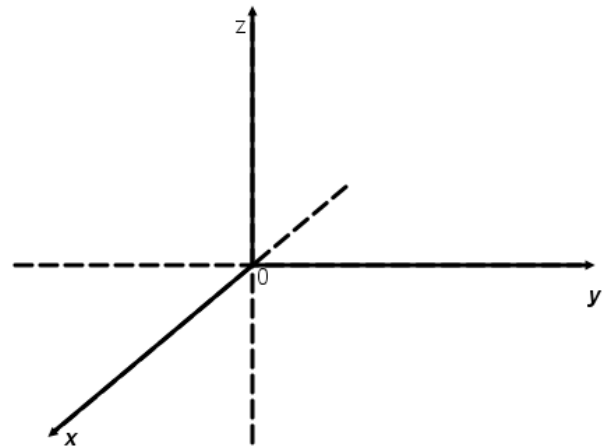
(a)  $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$



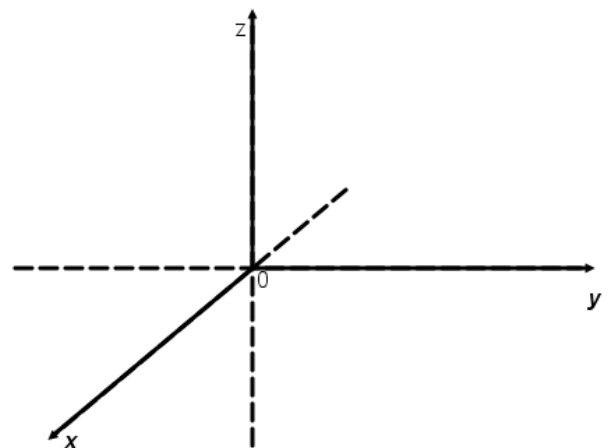
(b)  $\mathbf{r}(t) = \langle \cos at, \sin at, c \rangle$  where  $a$  and  $c$  are positive constants.



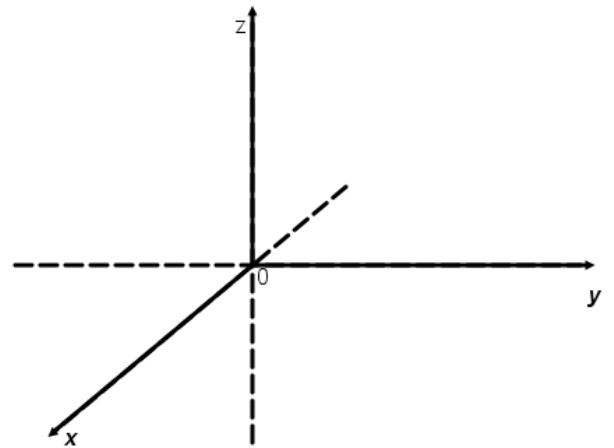
(c)  $\mathbf{r}(t) = \langle 2 \cos t, 3 \sin t, 1 \rangle, 0 \leq t \leq 2\pi$



(d)  $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$



(e)  $\mathbf{r}(t) = \langle 1 + t, 3 + 2t, 4 - 5t \rangle, -1 \leq t \leq 1.$



EXAMPLE 2. Show that the curve given by

$$\mathbf{r}(t) = \langle \sin t, 2 \cos t, \sqrt{3} \sin t \rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

## 13.2 Derivatives of Vector Functions

The derivative  $\mathbf{r}'$  of a vector function  $\mathbf{r}$  is defined just as for a real-valued function:

$$\frac{d\mathbf{r}(t_0)}{dt} = \mathbf{r}'(t_0) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h}$$

if the limit exists. The derivative  $\mathbf{r}'(t_0)$  is the tangent vector to the curve  $\mathbf{r}(t)$  at the point  $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle$ .

**THEOREM 3.** *If the functions  $x(t), y(t), z(t)$  are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

**EXAMPLE 4.** *Given  $\mathbf{r}(t) = (1 + t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$ .*

(a) *Find  $\mathbf{r}'(t)$*

(b) *Find a tangent vector to the curve at  $t = 0$ .*

(c) *Find a tangent line to the curve at  $t = 0$ .*

(c) *Find a tangent line to the curve at the point  $(1, 1, 0)$ .*