14.1: Functions of Several Variables

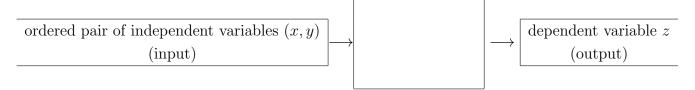
Consider the following formulas:

$$z = 2 - x - 4y \tag{1}$$

$$z = x^2 + y^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \tag{3}$$

$$z = \sqrt{1 - x^2 - y^2} \tag{4}$$



DEFINITION 1. Let $D \subset \mathbb{R}^2$. A function f of two variables is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by f(x, y).

The set D is the domain of f and the range of f is the set of values that f takes on, that is $\{f(x,y)|(x,y)\in D\}.$

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, x = f(y, z).

• GRAPH of f(x, y).

Recall that a graph of a function f of one variable is a curve C with equation y = f(x).

DEFINITION 3. The graph of f with domain D is the set:

$$S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \ (x, y) \in D\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation z = f(x, y).

EXAMPLE 4. Find the domain and sketch the graph of the functions (1)-(4). What is the range?

$$(1) \ z = 2 - x - 4y$$

$$D =$$

(2)
$$z = x^2 + y^2$$

$$D =$$

$$(3) z = \sqrt{x^2 + y^2}$$
$$D =$$

(4)
$$z = \sqrt{1 - x^2 - y^2}$$

 $D =$

$$D =$$

EXAMPLE 5. Sketch the domain of each of the following:

(a)
$$z = \sqrt{x} - \frac{5}{\sqrt{y}}$$



(b)
$$z = \ln(x^2 + \frac{y^2}{16} - 1)$$



• LEVEL (CONTOUR) CURVES method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. The level (contour) curves of a function of two variables are the curves with equations

$$f(x,y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

EXAMPLE 7. Sketch the level curves of the functions (2) and (3) for the values k = 0, 1, 2, 3, 4:

(2)
$$z = x^2 + y^2$$
 (3) $z = \sqrt{x^2 + y^2}$

• Functions of three variables.

DEFINITION 8. Let $D \subset \mathbb{R}^3$. A function f of three variables is a rule that assigns to each ordered pair (x, y, z) in D a unique real number denoted by f(x, y, z).

Examples of functions of 3 variables:

$$f(x, y, z) = x^{2} + y^{2} + z^{2},$$

$$u = xyz$$

$$T(s_{1}, s_{2}, s_{3}) = \ln s_{1} + 12s_{2} - s_{3}^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. Find the domain of the following function:

$$f(x,y,z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of f(x, y, z) remains fixed.

EXAMPLE 10. Find the level surfaces of the function $u = x^2 + y^2 - z$.

REMARK 11. For any function there exist a unique level curve (surface) through given point!!!