# 14.3: Partial Derivatives

DEFINITION 1. *If $f$ is a function of two variables, its* **partial derivatives** *are the functions $f_x$ and $f_y$ defined by*

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

*Conclusion:* $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change $x$ and hold $y$ fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change $y$ and hold $x$ fixed.

**Notations for partial derivatives:** If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y =$$

*RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x,y)$:*

1. To find $f_x$, regard $y$ as a constant and differentiate $f(x,y)$ with respect to $x$.
2. To find $f_y$, regard $x$ as a constant and differentiate $f(x,y)$ with respect to $y$.

EXAMPLE 2. *If $f(x,y) = x^3 + y^5 e^x$ find $f_x(0,1)$ and $f_y(0,1)$.*

EXAMPLE 3. *Find all of the first order partial derivatives for the following functions:*

**(a)** $z(x,y) = x^3 \sin(xy)$

**(b)** $u(x, y, z) = ye^{xyz}$

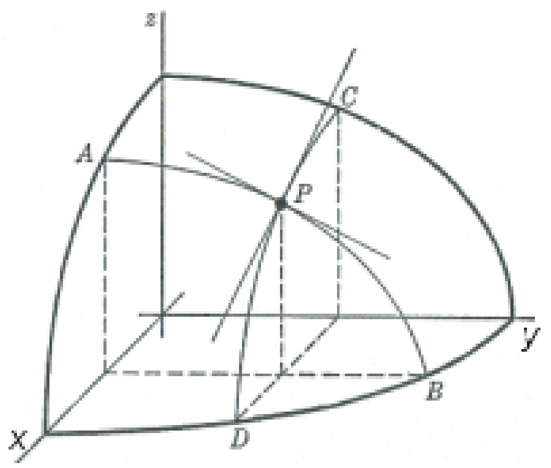
EXAMPLE 4. *The temperature at a point* $(x, y)$ *on a flat metal plate is given by*

$$T(x, y) = \frac{80}{1 + x^2 + y^2},$$

*where* $T$ *is measured in* $^\circ$C *and* $x, y$ *in meters. Find the rate of change of temperature with respect to distance at the point* $(1, 2)$ *in the* $y$*-direction.*

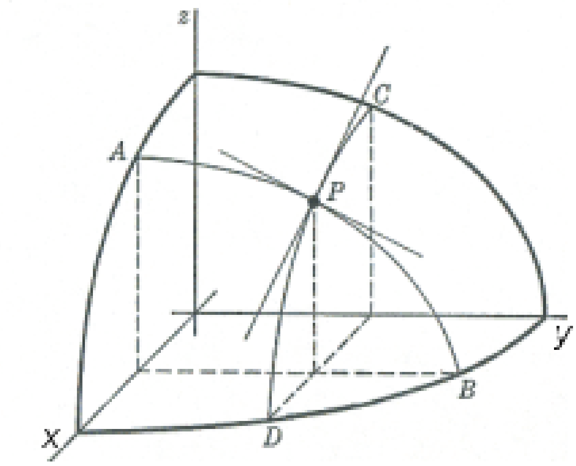
**Geometric interpretation of partial derivatives:** Partial derivatives are the *slopes of traces*:

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point $(a,b)$.




- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at $(a,b)$.

EXAMPLE 5. *If* $f(x,y) = \sqrt{4 - x^2 - 4y^2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*




**Higher derivatives:** Since both of the first order partial derivatives for $f(x,y)$ are also functions of $x$ and $y$, so we can in turn differentiate each with respect to $x$ or $y$. We use the following notation:

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}\\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}\\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}\\
(f_y)_y &= \quad = \quad = \quad = \quad =
\end{aligned}
$$

EXAMPLE 6. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

**Clairaut's Theorem.** *Suppose $f$ is defined on a disk $D$ that contains the point $(a,b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$ then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions $f_{yyx}$, $f_{xyy}$ and $f_{yxy}$ are continuous then

EXAMPLE 7. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

**(a)** $f_{xy}$

**(b)** $f_{zxy}$