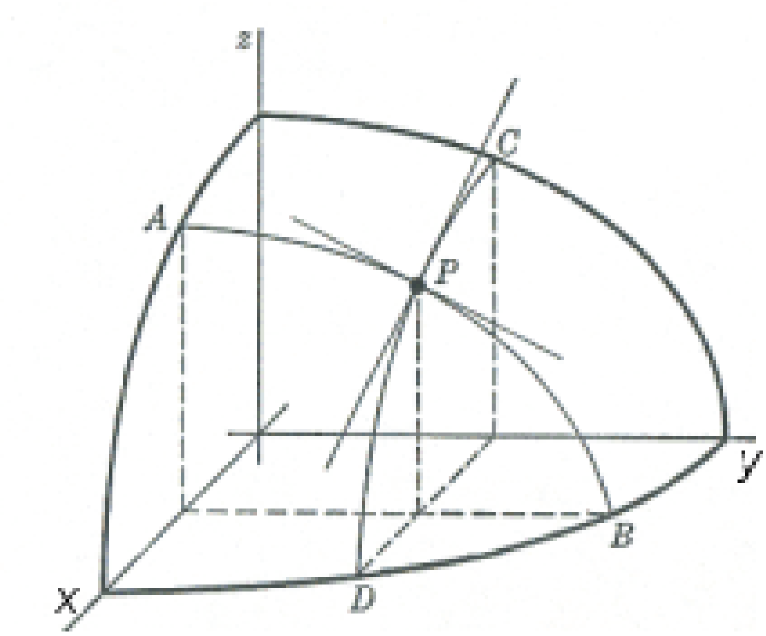
14.4: Tangent Planes and Linear Approximation

Tangent Plane to the graph of $f(x, y)$

Suppose that $f(x, y)$ has continuous first partial derivatives and a surface S has equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x, y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x, y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x, y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



THEOREM 1. *An equation of the tangent plane to the graph of the function $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

CONCLUSION: A normal vector to the tangent plane to the surface given by the equation $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \langle \qquad , \qquad , \qquad \rangle .$$

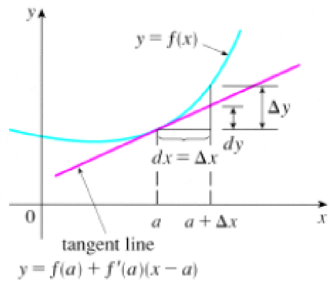
The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to the above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. It follows that this normal line can be expressed parametrically as

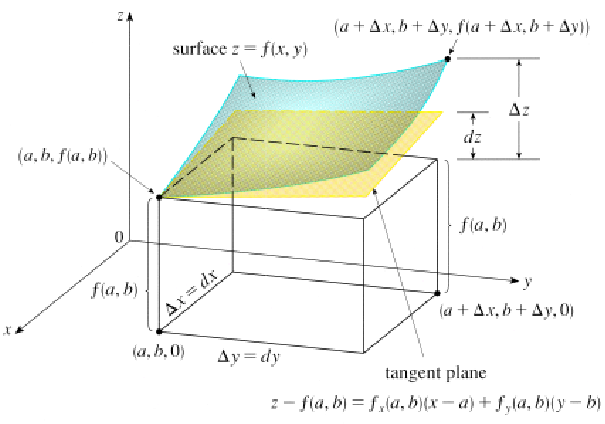
EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function* $z = x^2 + y^2 + 8$ *at the point* $(1, 1)$.

EXAMPLE 3. *Find parametric equations for the normal line to the surface* $z = e^{4y}\sin(4x)$ *at the point* $P(\pi/8, 0, 1)$

Differentials. Given $z = f(x, y)$. If Δx and Δy are given increments of $x = a$ and $y = b$ respectively, then the corresponding **increment** of z is

$$\Delta z(a, b) = f(a + \Delta x, b + \Delta y) - f(a, b). \quad (1)$$





[1]

[1]the pictures are from our textbook

The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **total differential**) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$

FACT: $\Delta z \approx \mathrm{d}z$.

This implies:

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + \mathrm{d}z(a,b)$$

or

EXAMPLE 4. *Use differentials to find an approximate value for* $\sqrt{1.03^2 + 1.98^3}$.

If $u = f(x,y,z)$ then the differential $\mathrm{d}u$ at the point $(x,y,z) = (a,b,c)$ is defined in terms of the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ of the independent variables:

$$\mathrm{d}u(a,b,c) = f_x(a,b,c)\mathrm{d}x + f_y(a,b,c)\mathrm{d}y + f_z(a,b,c)\mathrm{d}z.$$

EXAMPLE 5. *The dimensions of a closed rectangular box are measured as* 80 *cm,* 60 *cm and* 50 *cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*

A function $f(x, y)$ *is* **differentiable** *at* (a, b) *if its partial derivatives* f_x *and* f_y *exist and are continuous at* (a, b).

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a **linear function** of two variables.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$

The function $L(x, y)$ is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the tangent plane approximation of f at (a, b).