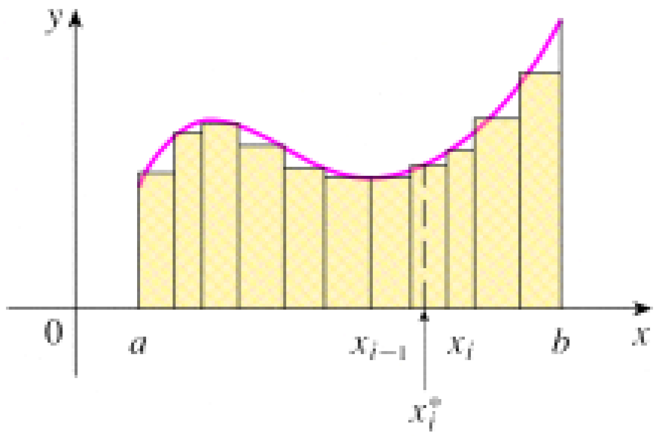
15.1: Double integrals over rectangles

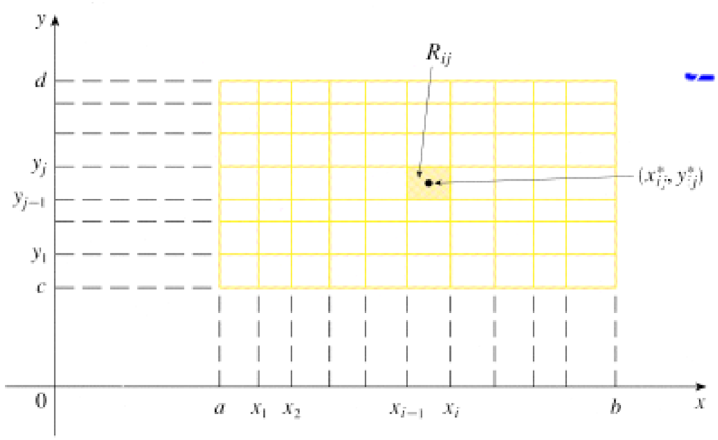
Recall that a single definite integral can be interpreted as **area**:



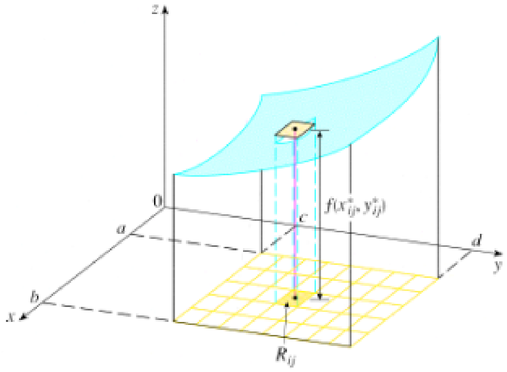
The exact area is also the definition of the definite integral:
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

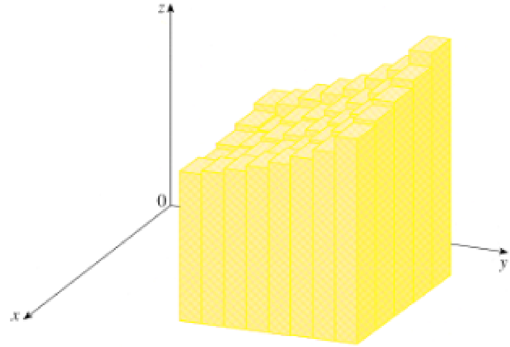
Problem: Assume that $f(x,y)$ is defined on a closed rectangle $R=[a,b]\times[b,c]=\{(x,y)\in\mathbb{R}^2 | a\le x\le b, c\le y\le d\}$ and $f(x,y)\ge 0$ over R. Denote by S the part of the surface $z=f(x,y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

Solution: Approximate the volume. Divide up $a\le x\le b$ into n subintervals and divide up $c\le y\le d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).



Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i*, y_j^*)$.





The volume is given by

$$\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x \Delta y$$

which is also the definition of a double integral

$$\iint_R f(x,y) \mathrm{d}A.$$

Another notation: $\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$

THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x,y) \geq 0$ and f is continuous on the rectangle $R = [a,b] \times [c,d]$, then the volume V of the solid S that lies above R and inder the graph of f, i.e.*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 | (x,y) \in R, 0 \leq z \leq f(x,y), (x,y) \in R\right\},$$

is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$

where $R = [-1,0] \times [-3,3]$ by identifying it as a volume of a solid.