15.3: Double Integrals in Polar Coordinates

The **polar coordinate system** consists of:

- the **pole** (or origin) labeled O;
- the **polar axis** which is a ray starting at O (usually drawn horizontally to the right);

The **polar coordinates** (r, θ) of a point P:

- θ is the angle between the polar axis and the line OP (the angle is positive if measured in counter-clockwise direction from the polar axis);
- r is the distance from O to P.

EXAMPLE 1. *Plot the points whose polar coordinates are given:*
(a) $(1, \pi/3)$ **(b)** $(5, -\pi/2)$.

The connection between polar and Cartesian coordinates:

$\cos\theta =$ $\sin\theta =$

$x =$ $y =$

$r^2 =$ $\tan\theta =$

REMARK 2. In converting from the Cartesian to polar coordinates we must choose θ so that the point (r, θ) lies in the correct quadrant.

EXAMPLE 3. *What curve is represented by the following polar equation*
(a) $r = 12$ *(b)* $\theta = \frac{\pi}{3}$

EXAMPLE 4. *Sketch the region in the Cartesian plane consisting of points whose polar coordinates satisfy the following conditions:* $1 \leq r \leq 2, \quad \pi/4 \leq \theta \leq \pi$.

EXAMPLE 5. *Find a polar equation for the curve represented by the given Cartesian equation:*

(a) $x^2 + y^2 = 2by$

(b) $(x - a)^2 + y^2 = a^2$

Using polar coordinates to evaluate double integrals

EXAMPLE 6. *Evaluate*

$$I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$$

where $D = \left\{(x, y) | \ 1 \le x^2 + y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}$.

THEOREM 7. *Change to polar coordinates in a double integral: Let* f *be a continuous on the region* D*. Denote by* D^* *the region representing* D *in the polar coordinates* (r, θ)*. Then*

$$\iint_D f(x, y) \, \mathrm{d}A = \iint_{D^*} f(r \cos\theta, r \sin\theta) \ r \, \mathrm{d}r \mathrm{d}\theta.$$

REMARK 8. Be careful not to forget the additional factor r on the right side of the formula.

Solution of Example 6:
Evaluate $I = \iint_D \arctan \frac{y}{x}\, \mathrm{d}A$, where $D = \left\{(x,y)|\ 1 \le x^2+y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}$.

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$.*

EXAMPLE 10. *Find the area of the region inside the circle* $r = 4\sin\theta$ *and outside the circle* $r = 2$.