

15.9: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

$$\int_a^b f(g(x))g'(x) dx = \int_\alpha^\beta f(u) du.$$

- conversion to polar coordinates:

$$\iint_D f(x, y) dA = \iint_{D^*} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

- conversion to cylindrical coordinates:

$$\iiint_E f(x, y, z) dV = \iiint_{E^*} f(r \cos \theta, r \sin \theta, z) r dr dz d\theta$$

- conversion to spherical coordinates:

$$\iiint_E f(x, y, z) dV = \iiint_{E^*} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

We call the equations that define the change of variables a **transformation**:

$$x = x(u, v), \quad y = y(u, v).$$

EXAMPLE 1. Determine the new region that we get by applying the transformation $x = 3u, y = u/2$ to the region $D = \left\{ (x, y) \mid \frac{x^2}{36} + y^2 = 1 \right\}$.

DEFINITION 2. The **Jacobian** of the transformation $x = x(u, v), y = y(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. Compute the Jacobian of the transformation $x = r \cos \theta, y = r \sin \theta$.

Change of variables for a double integral:

$$\iint_D f(x, y) \, dA = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

EXAMPLE 4. Evaluate

$$\iint_D e^{\frac{y-x}{y+x}} \, dA$$

where D is triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$.

EXAMPLE 5. Find mass of a lamina that occupies the region

$$D = \{(x, y) | 16 \leq x^2 + y^2 \leq 25, \quad 1 \leq x^2 - y^2 \leq 9, \quad x \geq 0, y \geq 0\}$$

with density $\rho(x, y) = 8xy$.