16.3: The fundamental Theorem for Line Integrals

16.4: Green's Theorem

DEFINITION 1. *A vector field* $\mathbf{F}$ *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* $\mathbf{F}$.

Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

where F' *is continuous on* $[a, b]$.

- **The fundamental Theorem for Line Integrals:** *Let* C *be a smooth curve given by* $\mathbf{r}(t)$, $a \leq t \leq b$. *Let* f *be a differentiable function of two or three variables and* ∇f *is continuous on* C. *Then*

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

REMARK 2. If C is a closed curve then

COROLLARY 3. *If* F *is a conservative vector field and* C *is a curve with initial point* A *and terminal point* B *then:*

EXAMPLE 4. *Find the work done by the gravitational field*

$$\mathbf{F}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z\rangle$$

in moving a particle with mass m *from the point* $(1, 2, 2)$ *to the point* $(3, 4, 12)$ *along a piecewise-smooth curve* C.

Notations And Definitions:

DEFINITION 5. *A piecewise-smooth curve is called a* **path**.

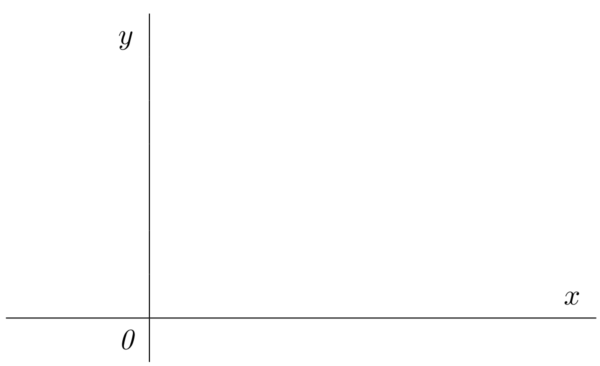
- **Types of curves:**

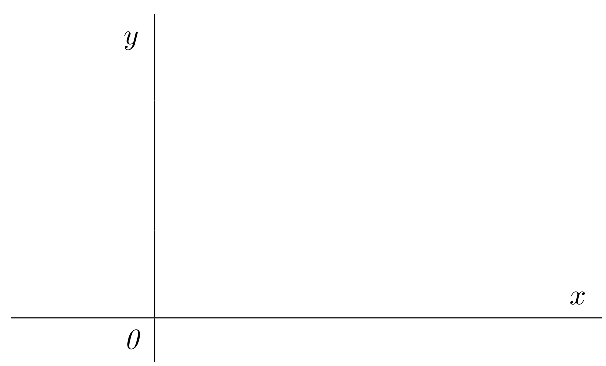
simple not closed	*not simple not closed*	*simple closed*	*not simple, closed*

- **Types of regions:**

simply connected	*not simply connected*

- **Convention:** *The* **positive orientation** *of a simple closed curve C refers to a single counter-clockwise traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \le t \le b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.*





- *The positively oriented boundary curve of D is denoted by ∂D.*

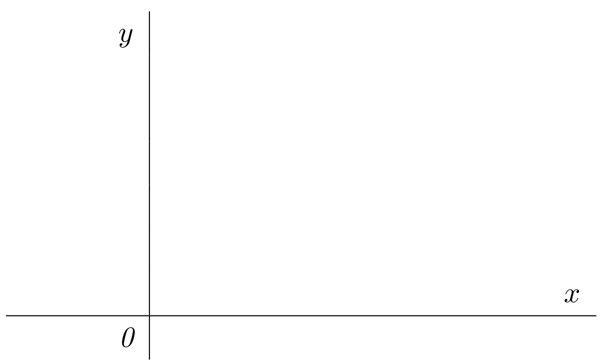
•**GREEN's THEOREM**: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A.$$

EXAMPLE 6. *Evaluate:*

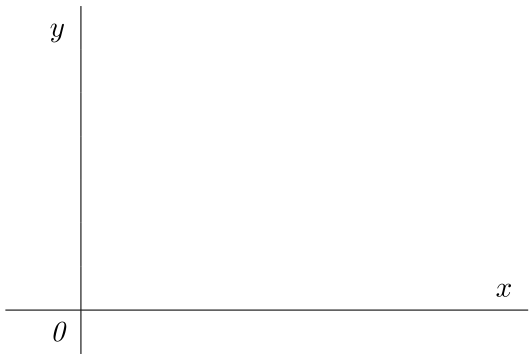
$$I = \oint_C e^x(1-\cos y)\,\mathrm{d}x - e^x(1-\sin y)\,\mathrm{d}y$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



EXAMPLE 7. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to (5,0), from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integral:*

$$I_1 = \oint_C (x^2y + \frac{1}{2}y^2)\,\mathrm{d}x + (xy + \frac{1}{3}x^3 + 3x)\,\mathrm{d}y$$



SUMMARY: Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on an open simply connected domain D. Suppose that P and Q have continuous partial derivatives through D. Then the facts below are equivalent.

The field $\mathbf{F}$ is **conservative**on D $\iff$ There exists f s.t. $\nabla f = \mathbf{F}$

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\int_{\breve{AB}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path in D

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C in D

EXAMPLE 8. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x, y) = \left\langle x^2 + y^2, 2xy \right\rangle$.

(b) $\mathbf{F}(x, y) = \left\langle x^2 + 3y^2 + 2, 3x + ye^y \right\rangle$

EXAMPLE 9. *Given* $\mathbf{F}(x, y) = \sin y\mathbf{i} + (x\cos y + \sin y)\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3, 0)$ *to the point* $(0, \pi/2)$.

(d) *Evaluate* $\oint_C \mathbf{F}\,\mathrm{d}\mathbf{r}$ *where* C *is an arbitrary path in* $\mathbb{R}^2$.

EXAMPLE 10. *Given*

$$\mathbf{F} = \left\langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \right\rangle.$$

Find a function f *s.t.* $\nabla f = \mathbf{F}$