16.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the **curl** of $\mathbf{F}$ is the *vector field* on $\mathbb{R}^3$ defined by

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

EXAMPLE 1. *Find the curl of the vector field*

$$\mathbf{F}(x,y,z) = \langle xy, x^2, yz \rangle .$$

Question What is the curl of a two-dimensional vector field

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} \ ?$$

Answer:

CONCLUSION: Green's Theorem in vector form:

$$\oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} =$$

THEOREM 2. *If a function $f(x, y, z)$ has continuous partial derivatives of second order then*

$$\text{curl}(\nabla f) = \mathbf{0}.$$

Proof:

COROLLARY 3. *If* $\mathbf{F}$ *is conservative, then* $\text{curl}\mathbf{F} = \mathbf{0}$.

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\text{curl}\mathbf{F} = 0$, *then* $\mathbf{F}$ *is a conservative vector field.*

EXAMPLE 5. *Let* $\mathbf{F}(x, y, z) = \langle x^9, y^9, z^9 \rangle$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$.

(c) *Evaluate* $\displaystyle\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F} \cdot d\mathbf{r}$.

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of F** is the *scalar field* on defined by

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \qquad\qquad .$$

EXAMPLE 6. *Find the divergence of the vector field*

$$\mathbf{F}(x,y,z) = \left\langle \sin(xyz), x^2, yz \right\rangle.$$

THEOREM 7. *If the components of a vector field* $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ *has continuous partial derivatives of second order then*

$$\text{div curl } \mathbf{F} = 0.$$

Proof.

EXAMPLE 8. *Is there a vector field* $\mathbf{G}$ *on* $\mathbb{R}^3$ *s.t.* curl $G = \langle yz, xyz, zy \rangle$*?*