16.7: Surface Integrals

Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

Solution:

If S is given by $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, $(u, v) \in D$, then the **surface integral of f over the surface S** is:

$$\iint_S f(x, y, z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u, v))|\mathbf{N}(u, v)|\,\mathrm{d}A =$$

EXAMPLE 1. *Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2+y^2 \leq 2x$, if its density is a function $\rho(x, y, z) = x^2+y^2+z^2$.*

• **Oriented surfaces.** We consider only two-sided surfaces.

Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x, y, z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.

If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x, y, z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:

`Convention:` *For closed surfaces the positive orientation is outward.*

- **Surface integrals of vector fields.**

DEFINITION 2. *If* $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\hat{\mathbf{n}}$, *then the* **surface integral of F over** S *is*

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S.$$

This integral is also called the **flux** *of* $\mathbf{F}$ *across* S.

Note that if S is given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} =$$

and

$$\mathrm{d}\mathbf{S} =$$

Finally,

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} =$$

EXAMPLE 3. *Find the flux of the vector field*

$$\mathbf{F} = \left\langle x^2, y^2, z^2 \right\rangle$$

across the surface

$$S = \left\{ z^2 = x^2 + y^2, 0 \le z \le 2 \right\}.$$