16.9: The Divergence Theorem

Let E be a simple solid region with the boundary surface S (which is a closed surface.) Let S be positively oriented (i.e.the orientation on S is outward that is, the unit normal vector $\hat{\mathbf{n}}$ is directed outward from E).

	Boundary
plain region D	
surface S	
solid E	

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a continuous vector field on an open region that contains E. Then

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\,\mathrm{d}V.$$

Divergence is the tendency of the vector field to *diverge from/to move toward* the point ($\mathrm{div}\mathbf{F} > 0$ corresponds to expansion; $\mathrm{div}\mathbf{F} < 0$ corresponds to compression).

In $\mathrm{div}\mathbf{F} = const$, then

The Divergence Theorem says that the divergence is the outgoing/ingoing flux per volume.

EXAMPLE 1. *Let* $E = \{(x, y, z) : x^2 + y^2 \le R^2, 0 \le z \le H\}$. *Find the flux of the vector field* $\mathbf{F} = \langle 1 + x, 3 + y, z - 10 \rangle$ *over* ∂E.

REMARK 2. If $\mathbf{F} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$ then

EXAMPLE 3. *Evaluate* $I = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$ *if* S *is the boundary of*

(a) *ellipsoid* $E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$ *and* $\mathbf{F} =$;

(b) *an arbitrary simple solid region* E *and* F *is an arbitrary continuous vector field.*

EXAMPLE 4. *Let E be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate* $I = \iint_S \langle x^3, 2xz^2, 3y^2z \rangle \cdot \mathrm{d}\mathbf{S}$ *if*

(a) *S is the boundary of the solid E.*

(B) *S is the part of the paraboloid $z = 4 - x^2 - y^2$ between the planes $z = 0$ and $z = 4$.*

EXAMPLE 5. *Verify the Divergence Theorem for the vector field* $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ *and* S*, which is the surface of the region enclosed by the cylinder* $x^2 + y^2 = 1$ *and the planes* $z = 1$ *and* $z = 0$*.*