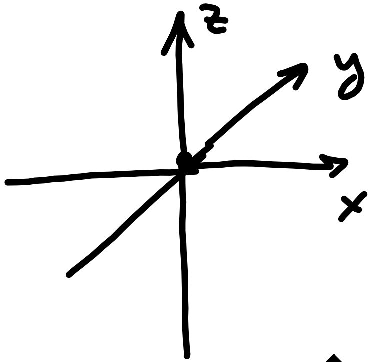
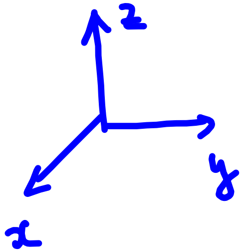
12 Vectors and Geometry of Space

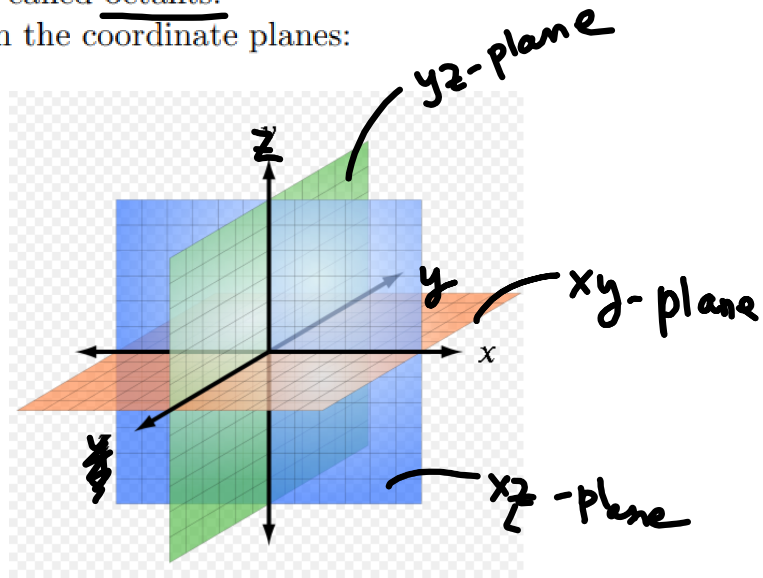
12.1: Three-dimensional Coordinate System

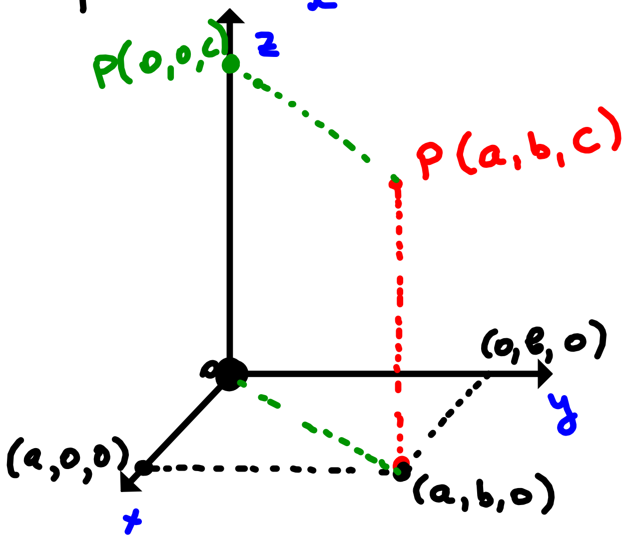
The three-dimensional coordinate system consists of the **origin** O and the **coordinate axes**: x-axis, y-axis, z-axis.The coordinate axes determine 3 **coordinate planes**: the xy-plane, the xz-plane and yz-plane. The coordinate planes divide space into 8 parts, called octants.

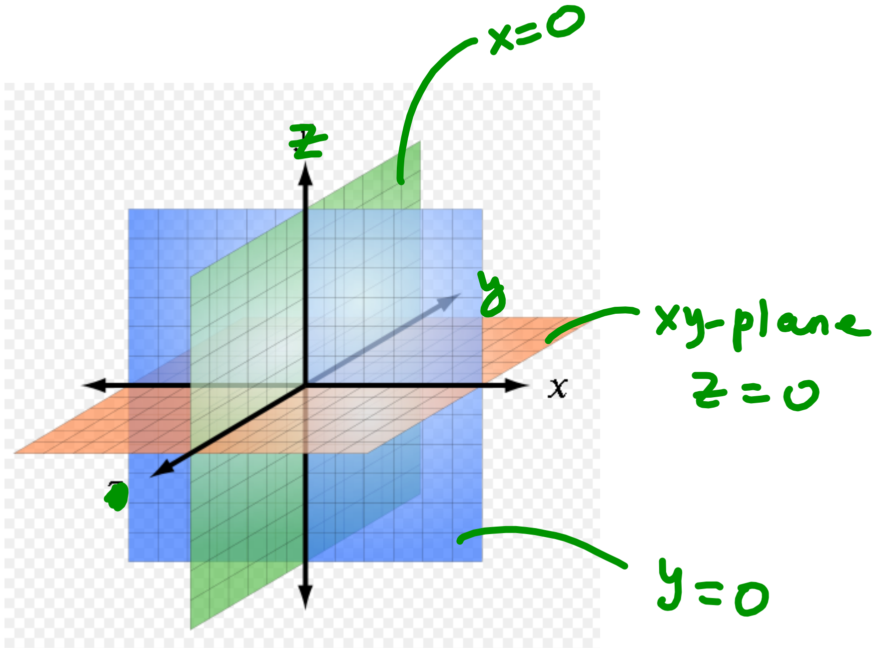
Representation of point $P(a, b, c)$ and its projections on the coordinate planes:










x=0
z
y
x
xy-plane
z=0
y=0

EXAMPLE 1. *Describe in words the regions of $\mathbb{R}^3$ represented by the following equation:*

(a) $z = 0$

$\underbrace{\text{The set}}_{\text{The geometric place}}$ of all points $(x, y, 0)$, or the xy- plane

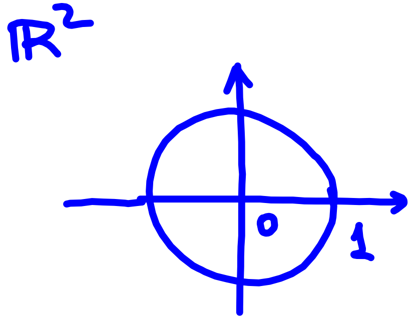
(b) $y = 0$

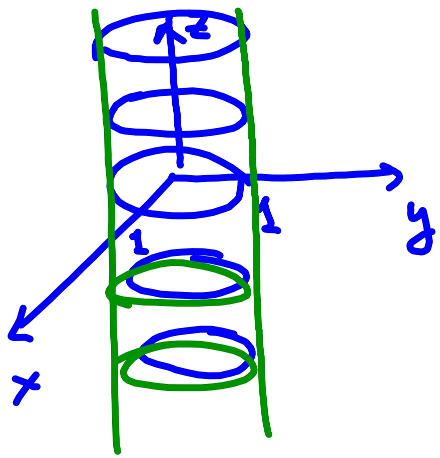
the set of all points $(x, 0, z)$, or the xz -plane

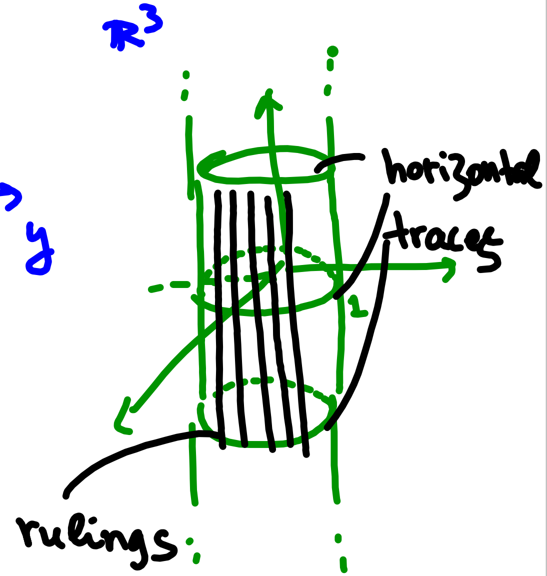
(c) $x = 0$ the set of all points $(0, y, z)$, or the yz -plane

Note that in $\mathbb{R}^2$ the graph of the equation involving x and y is a curve. In $\mathbb{R}^3$ an equation in x, y, z represents a **surface**.(It does not mean that we can't graph curves in $\mathbb{R}^3$.)

EXAMPLE 2. *Sketch the graph of* $x^2 + y^2 - 1 = 0$ *in* $\mathbb{R}^2, \mathbb{R}^3$.







a given plane curve.

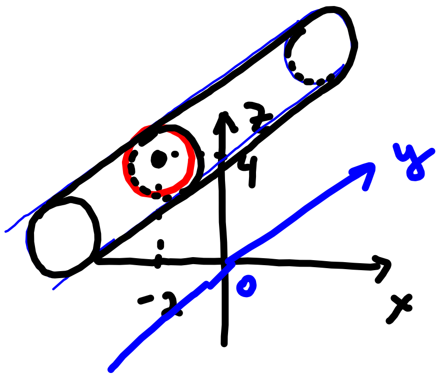
An equation that contains only two of the variables x, y, z represents a **cylindrical surface** *in* $\mathbb{R}^3$.

How to graph cylindrical surface:

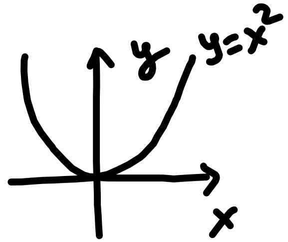
1. *graph the equation in the coordinate plane of the two variables that appear in the given equation;*

2. *translate that graph parallel to the axis of the missing variable.*

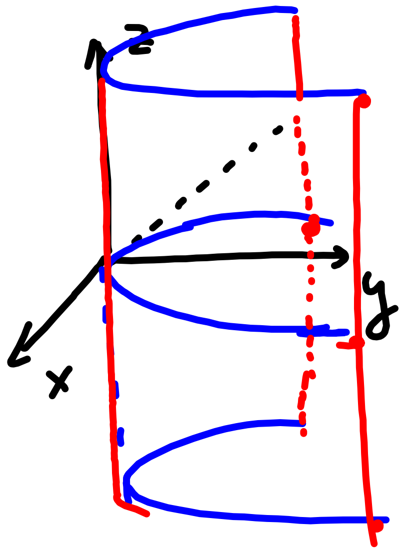
EXAMPLE 3. *Sketch the graph of* $(x+2)^2 + (z-4)^2 = 1$ *in* $\mathbb{R}^3$



EXAMPLE 4. *Sketch the graph of $y = x^2$ in $\mathbb{R}^3$*



parabolic cylinder



EXAMPLE 5. *Let S be the graph of $x^2 + z^2 - 10z + 21 = 0$ in $\mathbb{R}^3$.*

(a) *Describe S.*

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$x^2 + z^2 - 10z + 21 = 0$

$x^2 + (z^2 - 2 \cdot 5 \cdot z + 25) - 25 + 21 = 0$

S: $\boxed{x^2 + (z-5)^2 = 4}$

S is a cylinder with radius 2 parallel to the y-axis

$(x - 0)^2 + (z - \dots)^2 = \square^2$

(b) *The intersection of S with the xz- plane is* a circle centered at $(0,0,5)$ and with radius 2 in the xz

$S: \begin{cases} x^2 + z^2 - 10z + 21 = 0 \\ y = 0 \end{cases}$

(c) *The intersection of S with the yz- plane is* two lines $(0,y,7)$ and $(0,y,3)$ where $y \in \mathbb{R}$

$S: \begin{cases} x^2 + z^2 - 10z + 21 = 0 \\ x = 0 \end{cases} \Rightarrow z^2 - 10z + 21 = 0$

$z = 7, 3$

(d) *The intersection of S with the xy- plane is* empty

$\begin{cases} x^2 + z^2 - 10z + 21 = 0 \\ z = 0 \end{cases}$

$x^2 + 21 = 0$

no solutions

$$\underbrace{(x,y,z) \quad \text{and} \quad (0,0,0)}_{\text{distance} = 1}$$

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = 1$$

$$x^2 + y^2 + z^2 = 1$$

Spheres

- **Distance formula in $\mathbb{R}^3$:** The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Sphere centered at (x_0, y_0, z_0) with radius R :

distance between (x, y, z) and (x_0, y_0, z_0) should be $= R$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = R$$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

EXAMPLE 6. *Show that the equation $x^2 + y^2 + z^2 + 2x - 4y + 8z + 17 = 0$ represents a sphere, and find its center and radius.*

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$x^2 + 2x + y^2 - 4y + z^2 + 8z + 17 = 0$$

$$(x^2 + 2\cdot x\cdot 1 + 1^2) - 1^2 + (y^2 - 2\cdot y\cdot 2 + 2^2) - 2^2 + z^2 + 2\cdot z\cdot 4 + 4^2 - 4^2 + 17 = 0$$

$$(x+1)^2 + (y-2)^2 + (z+4)^2 + 17 - 1 - 4 - 16 = 0$$

$$(x+1)^2 + (y-2) + (z+4)^2 = 4$$

center at $(-1, 2, -4)$
radius $= 2$

In general, completing the squares in

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

produces an equation of the form

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k$$

- If $k > 0$ then the graph of this equation is a surface of sphere centered at (a,b,c) and with radius $\sqrt{k}$.
- If $k = 0$, then the graph is point with coordinates (a,b,c)
- If $k < 0$ then: no graph. (no real solutions)

$$\underbrace{(x-a)^2}_{\geq 0} + \underbrace{(y-b)^2}_{\geq 0} + \underbrace{(z-c)^2}_{\geq 0} = 0 \Rightarrow \begin{cases} (x-a)^2 = 0 \\ (y-b)^2 = 0 \\ (z-c)^2 = 0 \end{cases} \Rightarrow \begin{matrix} x=a \\ y=b \\ z=c \end{matrix} \Rightarrow$$

$$\Rightarrow (a,b,c)$$

Regions in $\mathbb{R}^3$

EXAMPLE 7. *Describe the set of all points in $\mathbb{R}^3$ whose coordinates satisfy the following inequality:* $\underbrace{x^2 + y^2 < 16}_{\text{a surface}}$

Namely, vertical circular cylinder about the z-axis with radius 4.

EXAMPLE 8. *Describe the following region:* $\{(x, y, z) | 9 \leq x^2 + y^2 + z^2 \leq 16\}$

region between two concentric
spheres centered at oridin
with radii 3 and 4.
(including these spheres)