12.2 & 12.3: Vectors and the Dot Product

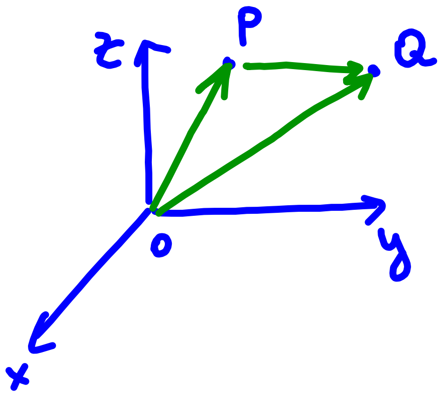
DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \overrightarrow{OQ} - \overrightarrow{OP}$$

The representation of the vector that starts at the point $O(0,0,0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.

point P $\longleftrightarrow$ position vector $\overrightarrow{OP}$



$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1,2,3)$ *and terminal point* $B(3,2,-1)$. *What is the position vector of the point* A?

$\overrightarrow{OA} = \boxed{\langle 1,2,3\rangle}$

BTW $\overrightarrow{OB} = \langle 3,2,-1\rangle$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 3-1, 2-2, -1-3\rangle = \boxed{\langle 2,0,-4\rangle}$

Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha \mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.

$\vec{a}$

$\alpha = 0$: $\alpha \vec{a} = \vec{0}$

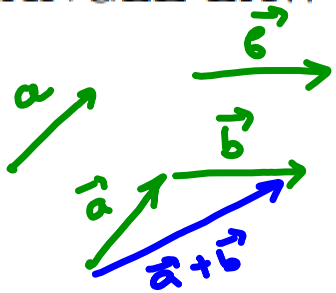
$|\alpha| < 1$: $\alpha > 0$, $\alpha\vec{a}$; $\alpha < 0$, $\alpha\vec{a}$

$|\alpha| = 1$: $\alpha\vec{a} = \begin{cases} \vec{a} \\ -\vec{a} \end{cases}$

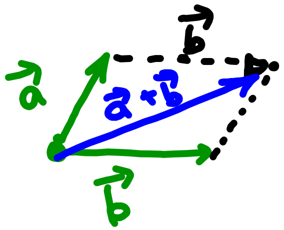
$|\alpha| > 1$

- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW



PARALLELOGRAM LAW



Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha \mathbf{a}$. Equivalently:

$$\underbrace{\mathbf{a} \| \mathbf{b} \Leftrightarrow \vec{b} = \alpha \vec{a}}_{\text{Rewrite in coordinates}} \quad \text{for some } \alpha \in \mathbb{R}$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \| \vec{b} \iff \langle b_1, b_2, b_3 \rangle = \alpha \langle a_1, a_2, a_3 \rangle$$

$$\iff \langle b_1, b_2, b_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$$

$$\iff \begin{cases} b_1 = \alpha a_1 \\ b_2 = \alpha a_2 \\ b_3 = \alpha a_3 \end{cases}$$

(If $a_1 a_2 a_3 \neq 0$, then $\vec{a} \| \vec{b} \iff \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} (= \alpha)$)

$\langle 1, 2, 3 \rangle \| \langle \frac{1}{3}, \frac{2}{3}, 1 \rangle$, because

$$\frac{\frac{1}{3}}{1} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

$$\langle 1, 2, 3 \rangle \nparallel \langle \frac{1}{5}, \frac{2}{5}, -\frac{3}{5} \rangle$$

$$\langle 0, 1, 2 \rangle \nparallel \langle 1, 0, 2 \rangle$$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0,0,0 \rangle$, $|\vec{0}| = 0$. $\vec{0} = \langle 0,0,0 \rangle$

Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector in the same direction as a:` $\boxed{\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}}$ *The process of multiplying a vectoe* $\mathbf{a}$ *by the reciprocal of its length to obtain a unit vector with the same direction is called* **normalizing a**.

Note that in $\mathbb{R}^2$ a nonzero vector $\mathbf{a}$ can be determined by its length and the angle from the positive x-axis:

$$|\hat{a}| = \left| \frac{\vec{a}}{|\vec{a}|} \right| = \left| \frac{1}{|\vec{a}|} \cdot \vec{a} \right| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$ a vector can be determined by its length and a vector in the same direction:

$$\boxed{\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}},}$$

i.e. $\mathbf{a}$ is equal to its length times a unit vector in the same direction.

EXAMPLE 3. *Find the components of a vector* **a** *of length* $\sqrt{5}$ *that extends along the line through the points* $M(2,5,0)$ *and* $N(0,0,4)$.

$|\vec{a}| = \sqrt{5}$

$\vec{a} \parallel \vec{MN}$

First find $\vec{MN}$

$\vec{MN} = \langle 0-2, 0-5, 4-0\rangle = \langle -2,-5,4\rangle$

Find unit vector/s collinear to $\vec{MN}$:

$$\pm\hat{a} = \pm\frac{\vec{MN}}{|\vec{MN}|} = \pm\frac{\langle -2,-5,4\rangle}{\sqrt{(-2)^2+(-5)^2+4^2}} = \pm\frac{\langle -2,-5,4\rangle}{3\sqrt{5}}$$

Use the above formula:

$$\vec{a} = |\vec{a}|\,\hat{a} = \sqrt{5}\,\frac{\pm\langle -2,-5,4\rangle}{3\sqrt{5}}$$

$$= \boxed{\pm\left\langle -\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}\right\rangle}$$

$$\pm\left\langle \frac{2}{3}, \frac{5}{3}, -\frac{4}{3}\right\rangle$$

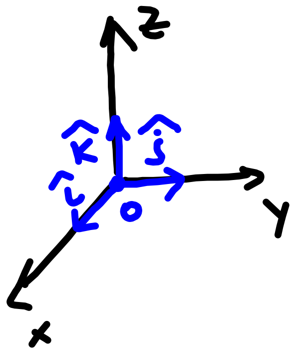
Standard Basis Vectors:

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$

unit vectors



$\hat{i} \perp \hat{j} \perp \hat{k}$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

EXAMPLE 4. *Given* $\mathbf{a} = \langle 1, 0, -3 \rangle$ *and* $\mathbf{b} = \langle 3, 1, 2 \rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}| = |\langle 3,1,2\rangle - \langle 1,0,-3\rangle| = |\langle 3-1, 1-0, 2-(-3)\rangle|$

$$= |\underbrace{\langle 2, 1, 5\rangle}_{\vec{b}-\vec{a}}| = \sqrt{2^2+1^2+5^2} = \sqrt{30}$$

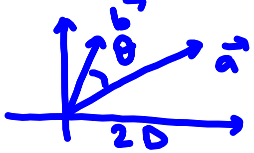
(b) *a unit vector that has the same direction as* $\vec{\mathbf{b}} - \vec{a}$.

$$\widehat{b-a} = \frac{\vec{b}-\vec{a}}{|\vec{b}-\vec{a}|} = \frac{\langle 2,1,5\rangle}{\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \left\langle \frac{2\sqrt{30}}{30}, \frac{\sqrt{30}}{30}, \frac{5\sqrt{30}}{30} \right\rangle$$

$$= \boxed{\left\langle \frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{30}, \frac{\sqrt{30}}{6} \right\rangle}$$

$\vec{a}\cdot\vec{b}$ is a scalar

Dot Product of two nonzero vectors **a** and **b** is the NUMBER:



$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta, \qquad (1)$$

geometric def.

where θ is the angle between **a** and **b**, $0 \leq \theta \leq \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a}\cdot\mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

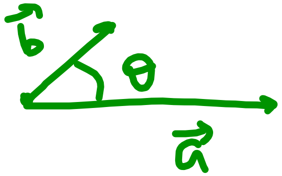
$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \qquad (2)$$

algebraic def

If θ is the *angle* between two nonzero vectors **a** and **b**, then

(1) $\Longrightarrow$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|} \overset{(2)}{=} \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$$



DEFINITION 5. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$. F.ex. $\hat{\imath} \perp \hat{\jmath} \perp \hat{k}$

EXAMPLE 6. *For two nonzero vectors* **a** *and* **b** *prove that*

(a)

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0$$

Proof $(\Rightarrow)$ $\vec{a} \perp \vec{b} \overset{\text{Def}}{\Rightarrow} \theta = \frac{\pi}{2} \overset{(1)}{\Rightarrow} \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos\frac{\pi}{2}}_{=0}$

$\Downarrow$

$\vec{a} \cdot \vec{b} = 0$

$(\Leftarrow)$ $\vec{a} \cdot \vec{b} = 0 \overset{(1)}{\Rightarrow} \underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cdot \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow$

$\Rightarrow \theta = \frac{\pi}{2}$ or $\underbrace{\theta = \frac{3\pi}{2}}_{\text{not in the range.}}$ $\Rightarrow \theta = \pi/2 \Rightarrow \vec{a} \perp b$

(b)

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

<u>Proof</u>

$$\sqrt{\vec{a}\cdot\vec{a}} \overset{(1)}{=} \sqrt{|\vec{a}|\cdot|\vec{a}|\cdot\cos\theta} \overset{\theta=0}{=}$$

$$= \sqrt{|\vec{a}|^2\cdot \underbrace{\cos 0}_{=1}} = \sqrt{|\vec{a}|^2} = |\vec{a}|.$$

EXAMPLE 7. *For what value(s) of c are the vectors $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ orthogonal?*

We know that $c\hat{i} + 2\hat{j} + \hat{k} \perp 4\hat{i} + 3\hat{j} + c\hat{k}$ if and only if

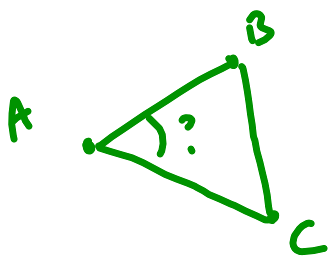
$$(c\hat{i} + 2\hat{j} + 1\hat{k}) \cdot (4\hat{i} + 3\hat{j} + c\hat{k}) = 0$$

$$4c + 2 \cdot 3 + 1 \cdot c = 0$$

$$5c = -6$$

$$\boxed{c = -\frac{6}{5}}$$

EXAMPLE 8. *The points* $A(6,-1,0)$, $B(-3,1,2)$, $C(2,4,5)$ *form a triangle. Find angle at* A.



$$\measuredangle A = \measuredangle(\vec{AB}, \vec{AC})$$

$$\vec{AB} = \langle -3-6, 1-(-1), 2-0 \rangle$$
$$= \langle -9, 2, 2 \rangle$$

$$\vec{AC} = \langle 2-6, 4-(-1), 5-0 \rangle$$
$$= \langle -4, 5, 5 \rangle$$

$$\vec{AB} \cdot \vec{AC} = \langle -9, 2, 2 \rangle \cdot \langle -4, 5, 5 \rangle$$
$$= -9 \cdot (-4) + 2 \cdot 5 + 2 \cdot 5$$
$$= 36 + 20 = 56$$

$$|\vec{AB}| = \sqrt{9^2 + 2^2 + 2^2} = \sqrt{89}$$

$$|\vec{AC}| = \sqrt{4^2 + 5^2 + 5^2} = \sqrt{66}$$

$$\cos \measuredangle A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{A}| \cdot |\vec{B}|}$$

$$= \frac{56}{\sqrt{89} \cdot \sqrt{66}}$$

$$\measuredangle A \approx 43^\circ$$