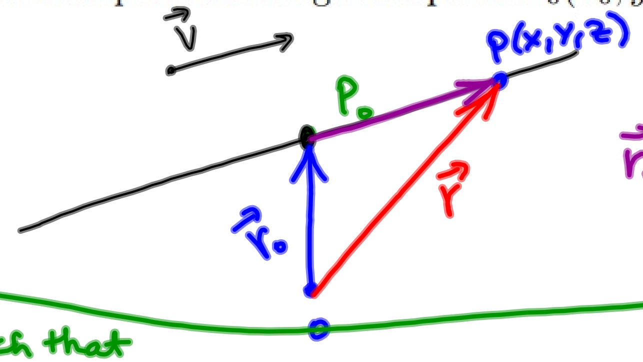
12.5: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\underbrace{\mathbf{v}}_{\text{direction vector}} = \langle a, b, c \rangle$.



$\vec{P_0P} \parallel \vec{v}$

$\vec{r}_0 + \vec{P_0P} = \vec{r}$

$P_0P = \underbrace{\vec{r} - \vec{r}_0}_{\langle x-x_0,\, y-y_0,\, z-z_0 \rangle \parallel \langle a,b,c \rangle} \parallel \vec{v}$

there is a constant t such that

$\vec{r} - \vec{r}_0 = t\vec{v}$

$\vec{r} = \vec{r}_0 + t\vec{v}$

given point

given direction vector

Vector equation	Parametric equations	Symmetric equations

$\mathbf{r}(t) = \overbrace{\langle x_0, y_0, z_0 \rangle}^{r_0} + t \overbrace{\langle a, b, c \rangle}^{\vec{v}}$,
$-\infty < t < \infty$
$x = x_0 + ta$
$y = y_0 + tb$
$z = z_0 + tc$
	$x = x_0 + at,$
	$y = y_0 + bt,$
	$z = z_0 + ct,$
	$-\infty < t < \infty$
		If $abc \neq 0$ then
		$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
		If, for example, $a = 0$ then
		the symmetric equations have the form:
		$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Vector equation	Parametric equations	Symmetric equations
$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$, ~~$-\infty < t < \infty$~~ $t \in \mathbb{R}$	$x = x_0 + at,$ $y = y_0 + bt,$ $z = z_0 + ct,$ ~~$-\infty < t < \infty$~~ $t \in \mathbb{R}$	If $abc \neq 0$ then $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ If, for example, $a = 0$ then the symmetric equations have the form: $x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$

EXAMPLE 1. *Complete the following.*

(a) *The equation* $\mathbf{r}(t) = \langle 1, 2, 3 \rangle + t \langle 4, 5, 6 \rangle$ *is a* vector *equation of the line passing through the point* (1,2,3) *and parallel to the vector* $\mathbf{v} =$ $\langle 4, 5, 6 \rangle$

(b) *The equation* $\mathbf{r}(t) = \langle 1, 2, 3 \rangle + t\mathbf{j}$ *is a* vector *equation of the line passing through the point* (1,2,3) *and parallel to the* y*-axis.*

(c) *The equations* $x = 2 - t, y = -t, z = 5$ *are* parametric *equations of the line passing through the point* (2,0,5) *and parallel to the vector* $\mathbf{v} =$ $\langle -1, -1, 0 \rangle$

(d) *The equations* $\frac{x-4}{5} = \underline{y+1} = \frac{z}{-3}$ *are* symmetric *equations of the line passing through the point* (4,-1, 0) *and parallel to the vector* $\mathbf{v} =$ $\langle 5, 1, -3 \rangle$

(e) *The equations* $\frac{x-4}{5} = y + 1, z = 2$ *are* Symmetric *equations of the line passing through the point* (4,-1 , 2) *and parallel to the vector* $\mathbf{v} =$ $\langle 5, 1, 0 \rangle$.

EXAMPLE 2. *Find vector equation of the line passing through the point* $(3, \overset{P_0}{-4}, 1)$ *and parallel to the vector* $\mathbf{v} = \langle 7, 0, -1 \rangle$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$ $\qquad r_0 = \overrightarrow{OP_0} = \langle 3, -4, 1 \rangle$

$\vec{r}(t) = \langle 3, -4, 1 \rangle + t \langle 7, 0, -1 \rangle, \ t \in \mathbb{R}$

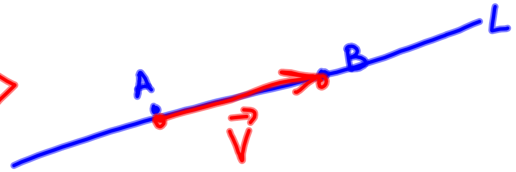
Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*
2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 3. *Consider the line L that passes through the points $A(1,1,1)$ and $B(2,3,-2)$.*

(a) *Find parametric equations of L.*



$\vec{v} = \overrightarrow{AB} = \langle 2-1, 3-1, -2-1 \rangle = \langle 1, 2, -3 \rangle$

$$x = 1 + t \cdot 1$$
$$y = 1 + t \cdot 2$$
$$z = 1 + t \cdot (-3)$$

or

$$\boxed{x = 1+t,\ y = 1+2t,\ z = 1-3t \ (t \in \mathbb{R})}$$

(b) *Find point C at that the L intersects the yz-plane.*

The yz-plane : $x = 0$

$0 = 1 + t \Rightarrow t = -1$

$y = 1 + 2t \qquad y = 1 + 2(-1) = -1$

$z = 1 - 3t \qquad z = 1 - 3\cdot(-1) = 4$

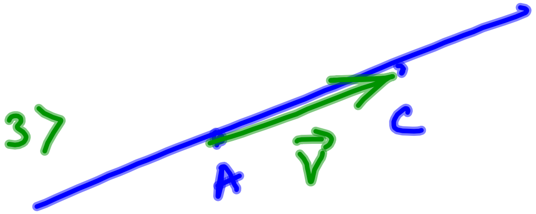
$\Rightarrow \boxed{C(0, -1, 4)}$

(c) *Find parametric equations describing the line segment joining the points A and C.*

$A = (1,1,1)$, $C\,(0,-1,4)$

Find line through A and C :

$\vec{v} = \vec{AC} = \langle 0-1, -1-1, 4-1 \rangle = \langle -1,-2,3 \rangle$



$x = 1-t$, $y = 1-2t$, $z = 1+3t$

If $t=0$, we get point A

$t=1$, we get point C

The segment from A to C :

$$\boxed{x = 1-t,\quad y = 1-2t,\quad z = 1+3t,\quad \text{where } 0 \le t \le 1}$$

EXAMPLE 4. *Determine whether the lines*

$\vec{v}_1 = \langle 1, 3, -1 \rangle$

$$L_1: \quad x - 1 = \frac{y+2}{3} = \frac{z-4}{-1}$$

and

$\vec{v}_2 = \langle 2, 1, 4 \rangle$

$$L_2: \quad x = 2t, \quad y = 3 + t, \quad z = -3 + 4t$$

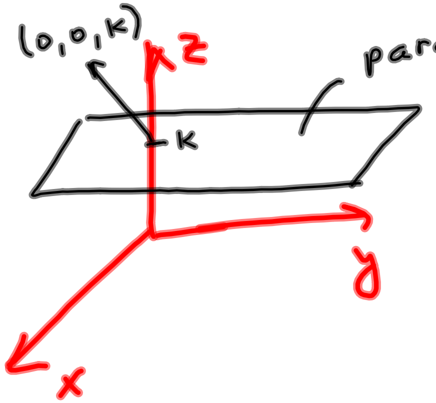
are parallel, skew, or intersecting.

$\frac{1}{2} \neq \frac{3}{1} \Rightarrow \vec{v}_1 \nparallel v_2 \Rightarrow L_1 \nparallel L_2$

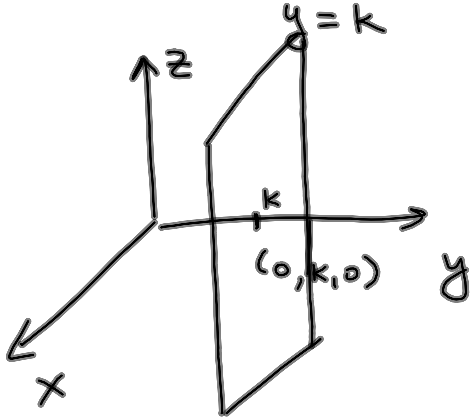
Planes

Planes parallel to the coordinate planes: $z=0$, $y=0$, $x=0$

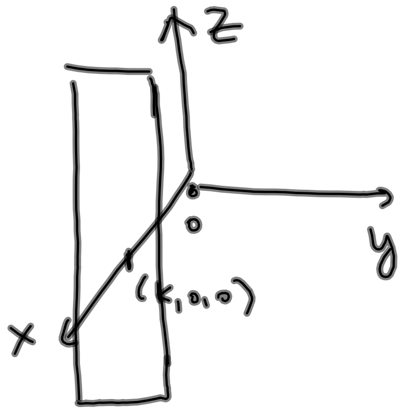
$z=0$, $z=k$



parallel to the xy-plane



x = k



Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $\boxed{P_0(x_0, y_0, z_0)}$ in the plane and a vector $\mathbf{n} = \langle a, b, c\rangle$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.

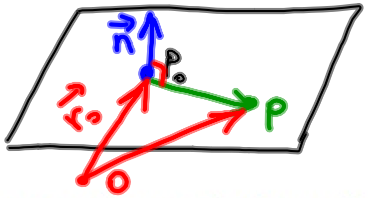
$\vec{r}_0 = \overrightarrow{OP_0}$

$\vec{r} = \overrightarrow{OP}$

$\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$

$\vec{n} \perp \overrightarrow{P_0P}$ OR

$\vec{n} \perp \vec{r} - \vec{r}_0 \iff \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$



Vector equation of the plane: $\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \mathbf{0}} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$

$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Scalar equation of plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 5. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

$\to x_0, y_0, z_0$

$P_0\ (1,2,1)$

$\vec{n} = \langle 2,3,4\rangle$

$$\boxed{2(x-1)+3(y-2)+4(z-1)=0}$$

$$2x-2+3y-6+4z-4=0$$

$$2x+3y+4z=12$$

The x-intercept: $y=z=0 \Rightarrow 2x=12 \Rightarrow x=6$

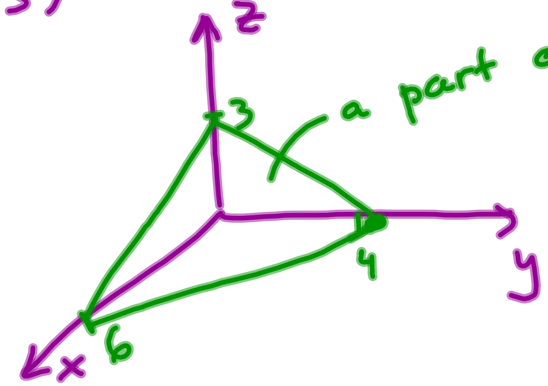
$(6,0,0)$

The y-intercept: $x=z=0 \Rightarrow 3y=12 \Rightarrow y=4$

$(0,4,0)$

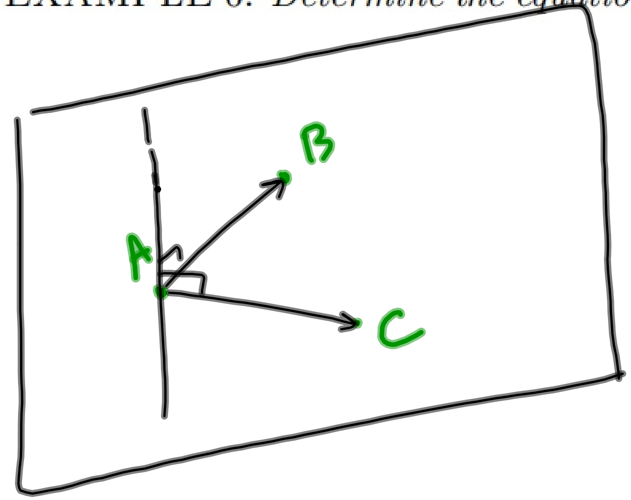
The z-intercept: $x=y=0 \Rightarrow 4z=12 \Rightarrow z=3$

$(0,0,3)$



a part of the plane in the first octant.

EXAMPLE 6. *Determine the equation of the plane through the points $A(1,1,1)$, $B(0,1,0)$ and $C(1,2,3)$.*



$$\overrightarrow{AB} = \langle 0-1,\ 1-1,\ 0-1 \rangle = \langle -1, 0, -1 \rangle$$

$$\overrightarrow{AC} = \langle 1-1,\ 2-1,\ 3-1 \rangle = \langle 0, 1, 2 \rangle$$

We can choose $\vec{n}$ as

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \hat{i} \cdot 1 \quad - \hat{j}(-2) + \hat{k}(-1)$$

$$= \langle 1, 2, -1 \rangle$$

$$1 \cdot (x-1) + 2(y-1) + (-1)(z-1) = 0$$ Scalar equation

$$x - 1 + 2y - 2 - z + 1 = 0$$

$$\boxed{x + 2y - z = 2}$$ linear equation

- Two planes are **parallel** if their normal vectors are parallel.
- Two planes are **orthogonal** if their normal vectors are orthogonal.
- If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 7. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

(a) *Find normal vectors corresponding to these planes.*

also

$\vec{n}_1 = \langle 2,3,1\rangle$ | $\langle 2,3,1\rangle$

$\vec{n}_2 = \langle -4,-6,-2\rangle$ | $\langle 2,3,1\rangle$

$\vec{n}_3 = \langle 2,0,-4\rangle$ | $\langle 1,0,-2\rangle$

$\vec{n}_4 = \langle -2,3,1\rangle$ | $\langle -2,3,1\rangle$

(b) *Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.*

(i) P_1 *and* P_2 parallel

$\vec{n}_1 \parallel \vec{n}_2 \Rightarrow P_1 \parallel P_2 \Rightarrow \measuredangle P_1, P_2 = 0$

(ii) P_1 *and* P_3 orthogonal

$\vec{n}_1 \cdot \vec{n}_3 = \langle 2,3,1\rangle \cdot \langle 2,0,-4\rangle = 4+0-4 = 0.$

$\measuredangle P_1, P_3 = \pi/2$

(iii) P_1 *and* P_4 neither

$$\cos \measuredangle P_1, P_4 = \cos \measuredangle \vec{n}_1, \vec{n}_4 = \frac{\vec{n}_1 \cdot \vec{n}_4}{|\vec{n}_1| \cdot |\vec{n}_4|}$$

$$= \frac{\langle 2,3,1\rangle \cdot \langle -2,3,1\rangle}{\left(\sqrt{2^2+3^2+1^2}\right)^2} = \frac{-4+9+1}{4+9+1}$$

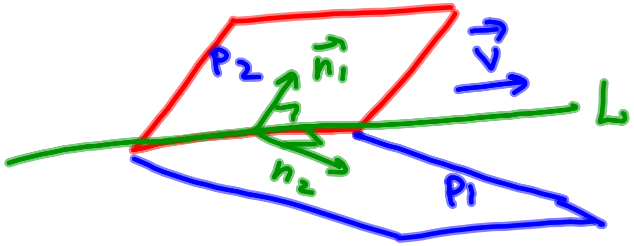
$$= \frac{6}{14} = \frac{3}{7}$$

$\measuredangle P_1, P_2 = \arccos \frac{3}{7} \approx \ldots$

Line as an intersection of two non parallel planes:

$$L : \begin{cases} P_1: a_1x + b_1y + c_1z + d_1 = 0 \\ P_2: a_2x + b_2y + c_2z + d_2 = 0 \end{cases} \qquad \begin{matrix} \vec{n}_1 = \langle a_1, b_1, c_1 \rangle \\ \vec{n}_2 = \langle a_2, b_2, c_2 \rangle \end{matrix}$$

The direction vector of L is $\vec{v} = \mathbf{n}_1 \times \mathbf{n}_2$.



$\vec{n}_1 \perp L$ and $\vec{n}_2 \perp L$

$\vec{n}_1 \perp \vec{v}$ and $\vec{n}_2 \perp \vec{v}$ $\Rightarrow$

$\Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2$

EXAMPLE 8. *Find an equation of the line given as intersection of two planes:*

$$\text{line } L: \begin{cases} P_1: & x - y + 3z = 0 \\ P_2 & x + y + 4z = 2 \end{cases} \qquad \vec{n}_1 = \langle 1, -1, 3 \rangle \quad \vec{n}_2 = \langle 1, 1, 4 \rangle$$

① direction vector for L:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = \hat{i}\,(-4-3) - \hat{j}\,(4-3) + \hat{k}\,(1-(-1)) = \langle -7, -1, 2 \rangle$$

② To find a point on L substitute, for example, $z=0$ into the plane equations:

$$+\begin{cases} x - y = 0 & \Rightarrow x = y \\ x + y = 2 \end{cases}$$

$$\left.\begin{array}{l} \\ 2x = 2 \Rightarrow x = 1 \end{array}\right\} \Rightarrow x = y = 1$$

$$P_0(1, 1, 0)$$

③ Line equation:

$$\begin{aligned} x &= 1 + t\cdot(-7) \\ y &= 1 + t\cdot(-1) \\ z &= 0 + t\cdot 2 \end{aligned} \qquad \text{OR}$$

$$\boxed{x = 1 - 7t,\ y = 1 - t,\ z = 2t \quad (t \in \mathbb{R})}$$