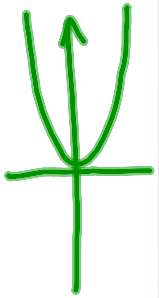
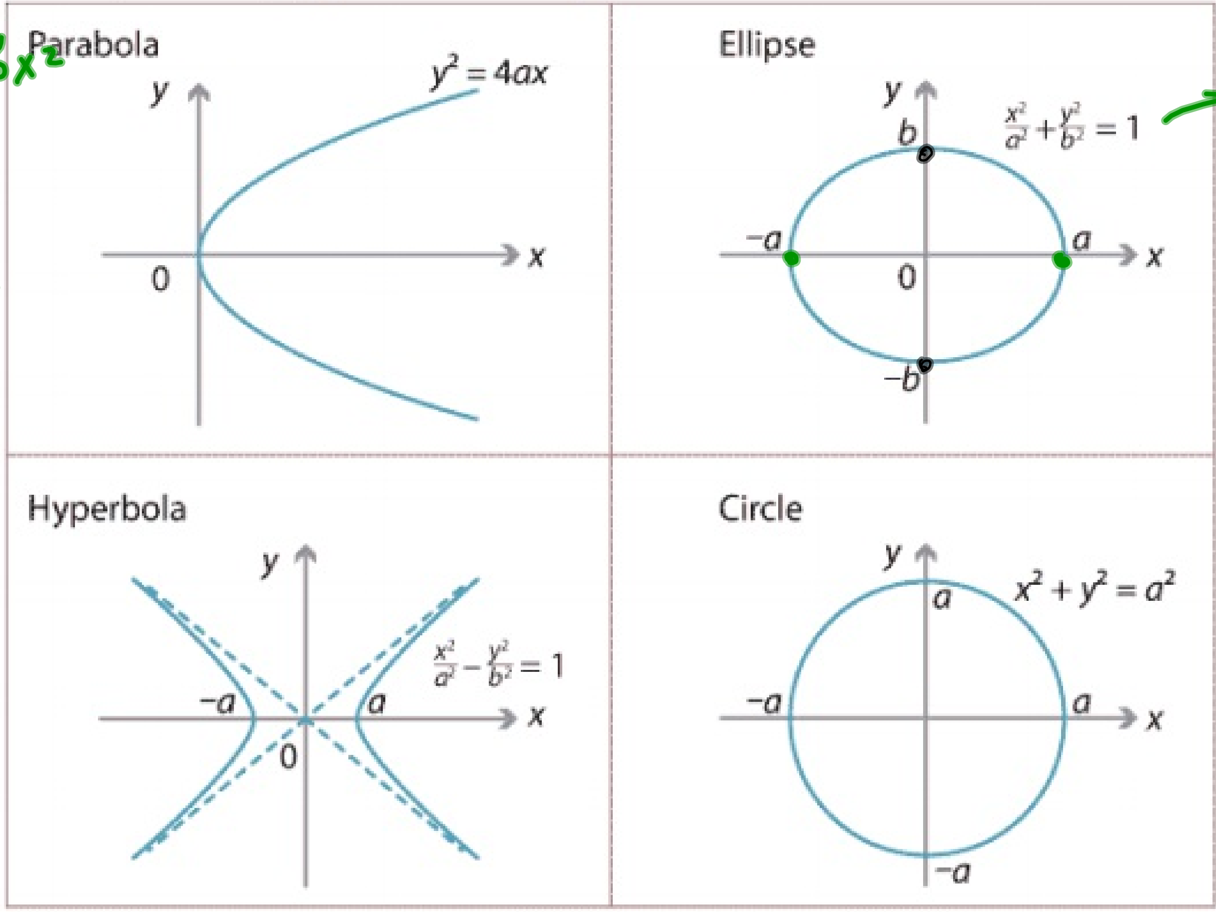
12.6: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.

$y = 6x^2$





$y = 0$

$\Downarrow$

$\frac{x^2}{a^2} = 1$

$\Downarrow$

$x = \pm a$

The most general second-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0, \tag{1}$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

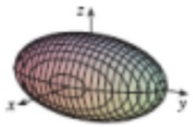
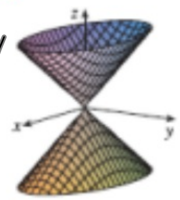
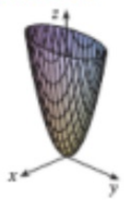
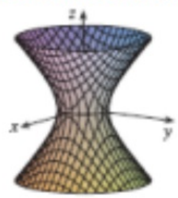
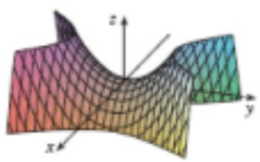
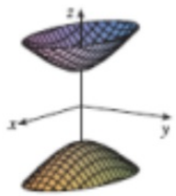
By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories: see sect. 1.1 in notes

ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders. (shown in the table below.)

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z=k$), yz-traces (by $x=0$) and xz-traces (by $y=0$).
3. Intercepts (in some cases).

To find the equation of a trace substitute the equation of the plane into the equation of the surface.

Note, in the examples below *the constants $a, b,$ and c are assumed to be positive.*

EXAMPLE 1. *Use traces to sketch the following quadric surfaces:*

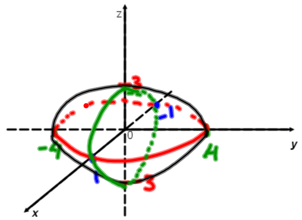
(a)

$$x^2 + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution

- *Find* **intercepts***:*
  - *x-intercepts: if $y = z = 0$ then $x =$* ± 1
    $(\pm 1, 0, 0)$
  - *y-intercepts: if $x = z = 0$ then $y =$* ± 4
    $(0, \pm 4, 0)$
  - *z-intercepts: if $x = y = 0$ then $z =$* ± 3
    $(0, 0, \pm 3)$

- *Obtain* **traces of***:*
  - *the xy-plane: plug in $z = 0$ and get $x^2 + \frac{y^2}{16} = 1$* ellipse
  - *the yz-plane: plug in $x = 0$ and get*
    $\frac{y^2}{16} + \frac{z^2}{9} = 1$
  - *the xz-plane: plug in $y = 0$ and get*
    $x^2 + \frac{z^2}{9} = 1$



  - *plug in $z = k$*
  - *plug in $x = k$*
  - *plug in $y = k$*

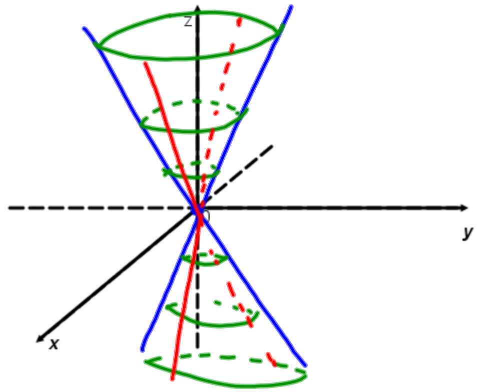
ellipses

(b)

$$z^2 = x^2 + \frac{y^2}{9}$$

Plane	*Trace*
$z = k$	$x^2 + \frac{y^2}{9} = k^2$ $k=0$ $(0,0,0)$ $k \neq 0$ ellipse
$x = 0$	$z^2 = \frac{y^2}{9}$ or $z = \pm \frac{y}{3}$ two lines through origin
$y = 0$	$z^2 = x^2$, or $z = \pm x$ two lines through the origin

elliptical cone

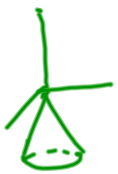


Note: If the horizontal traces are circles, for example,

$$z^2 = x^2 + y^2,$$

then we get a circular cone.

Also, $z = \sqrt{x^2+y^2}$ is 

$z = -\sqrt{x^2+y^2}$ is 

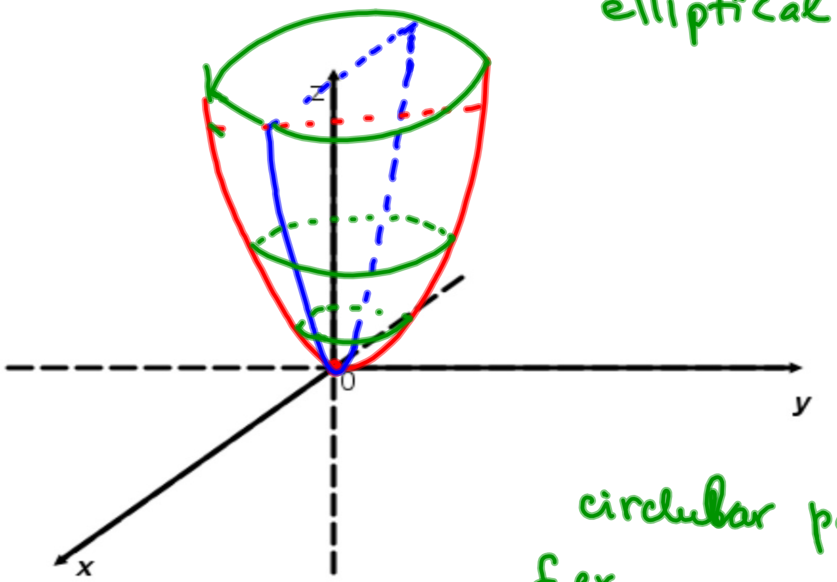
(c) horizontal traces

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

Plane	Trace
$z = k$	$\frac{x^2}{4} + \frac{y^2}{9} = k$ $k<0$ no points; $k=0$ $(0,0,0)$; $k>0$ ellipses
$x = 0$	$z = \frac{y^2}{9}$ parabola
$y = 0$	$z = \frac{x^2}{4}$ parabola

vertical traces

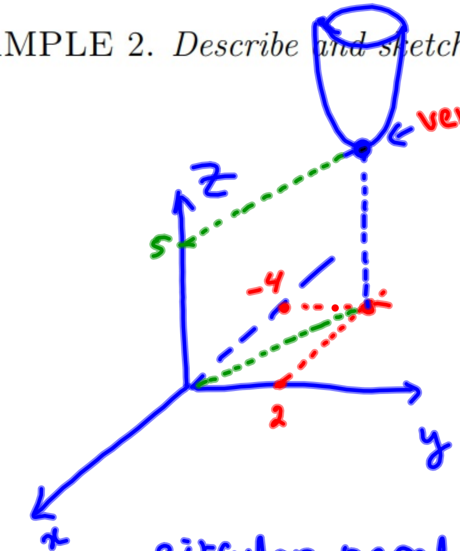
elliptical paraboloid



circular paraboloid

f.ex. $z = x^2 + y^2$

EXAMPLE 2. *Describe and sketch the surface* $z = (x+4)^2 + (y-2)^2 + 5$.



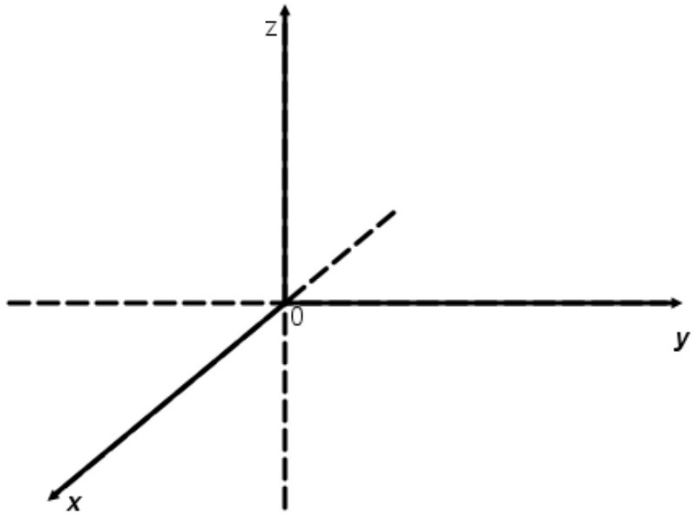
vertex $(-4, 2, 5)$

$$\underbrace{z-5}_{Z} = (\underbrace{x+4}_{X})^2 + (\underbrace{y-2}_{Y})^2$$

$$Z = X^2 + Y^2$$

vertex $(0,0,0)$

circular paraboloid with the vertex at $(-4, 2, 5)$



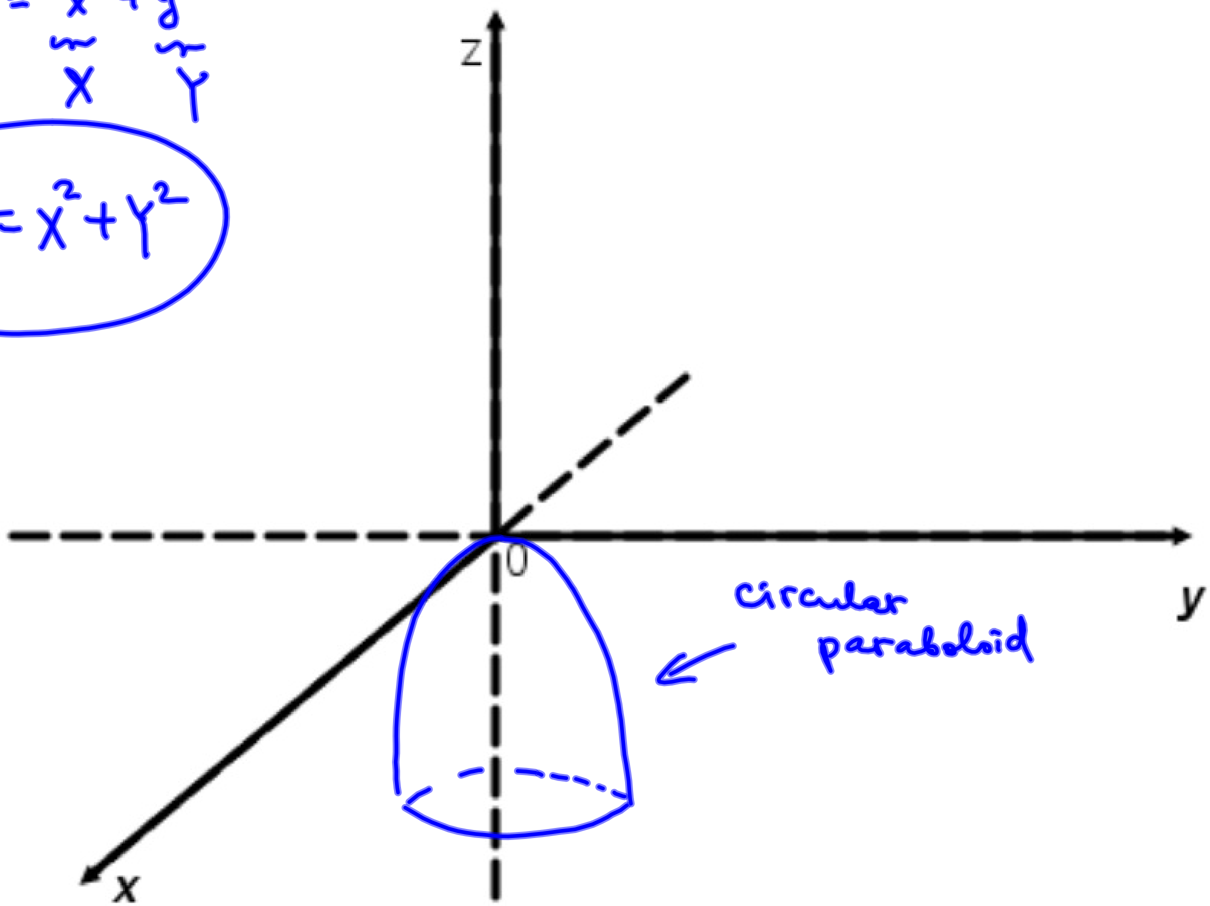
Note that **replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.**

EXAMPLE 3. *Identify and sketch the surface.*

(a) $z = -(x^2 + y^2)$

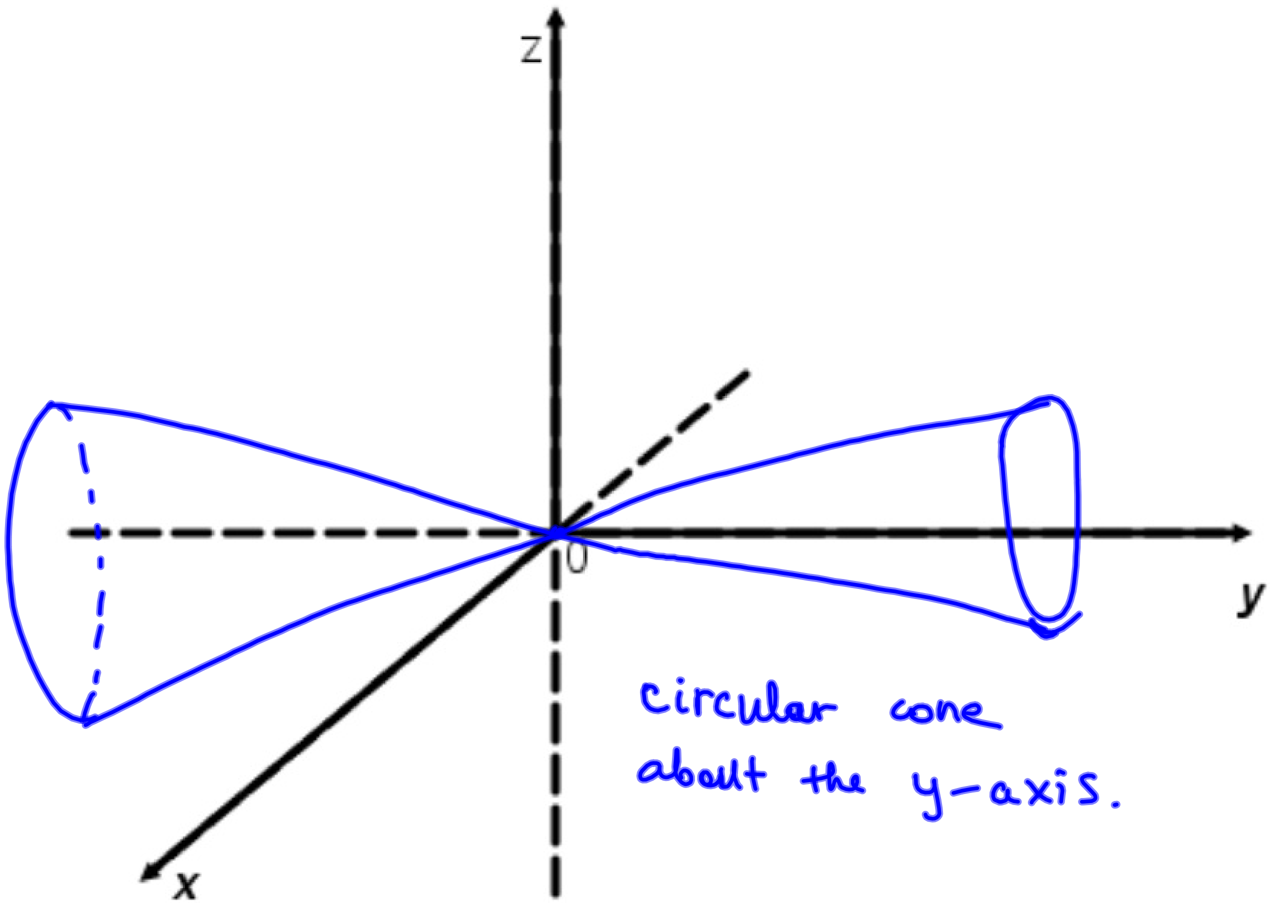
$\underbrace{-z}_{Z} = \underbrace{x}_{X}^2 + \underbrace{y}_{Y}^2$

$Z = X^2 + Y^2$



(b) $y^2 = x^2 + z^2$ → Use traces

OR ↙ $Z^2 = X^2 + Y^2$



circular cone about the y-axis.

$(a \pm b)^2 = a^2 + 2ab + b^2$

EXAMPLE 4. *Classify and sketch the surface*

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$

Complete squares

$$(x^2 - 4x + 4) - 4 + (y^2 - 6y + 9) - 9 + z + 13 = 0$$

$$(x-2)^2 + (y-3)^2 = -z$$

circular paraboloid with vertex at $(2,3,0)$ (see the picture).