13.1: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

Space curve is given by **parametric equations***:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \ \text{ in } \ I\},$$

where I is an interval and t is a **parameter***.*

FACT: Any continuous vector-function $\mathbf{r}(t)$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$.

Any parametric curve has a **direction of motion** *given by increasing of parameter.*

EXAMPLE 1. *Describe the curve defined by the vector function (indicate direction of motion):*

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$

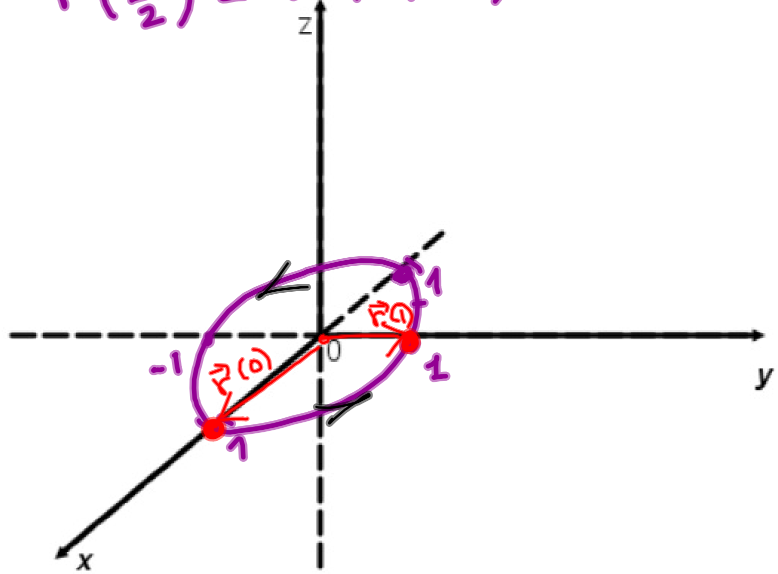
$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \xrightarrow{\text{eliminate parater}} \underbrace{x^2 + y^2 = 1}_{\text{cylinder}}$

$\underbrace{z = 0}_{\text{the xy plane}}$

We get line of intersection between cylinder $x^2+y^2=1$ and the xy-plane $(z=0)$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

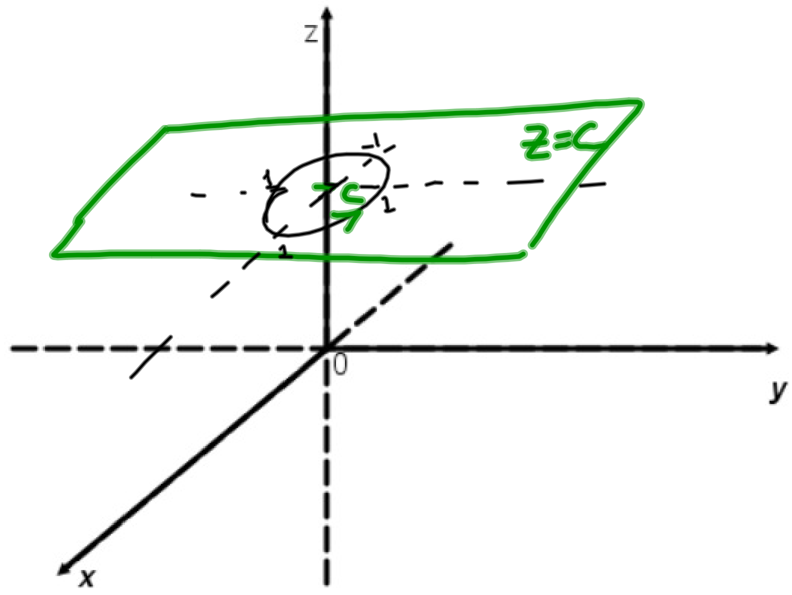
$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, 0 \rangle$



(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c \rangle$ *where a and c are positive constants.*

$$\left.\begin{array}{l} x = \cos at \\ y = \sin at \end{array}\right\} \quad \underbrace{x^2 + y^2 = 1}_{\text{cylinder}}$$

$$\underbrace{z = c}_{\text{plane}}$$



circle with radius 1
centered at $(0, 0, c)$
in the plane $z = c$

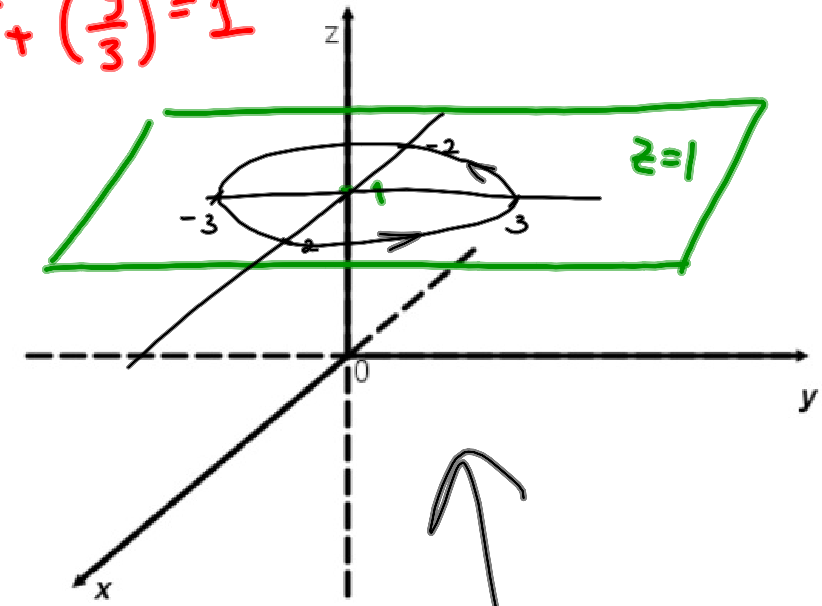
$$\vec{r}(0) = \langle 1, 0, c \rangle$$

$$\vec{r}\left(\frac{\pi}{2a}\right) = \langle 0, 1, c \rangle$$

(c) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1\rangle$, $0 \le t \le 2\pi$

$$\left.\begin{array}{l} x = 2\cos t \\ y = 3\sin t \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos t = \frac{x}{2} \\ \sin t = \frac{y}{3} \end{array}\right\} \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$ elliptical cylinder

$\underbrace{z = 1}_{\text{plane}}$



$\vec{r}(t)$ is line of intersection between the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the plane $z = 1$.

In other words, $\vec{r}(t)$ is ellipse in the plane $z = 1$.

(d) $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

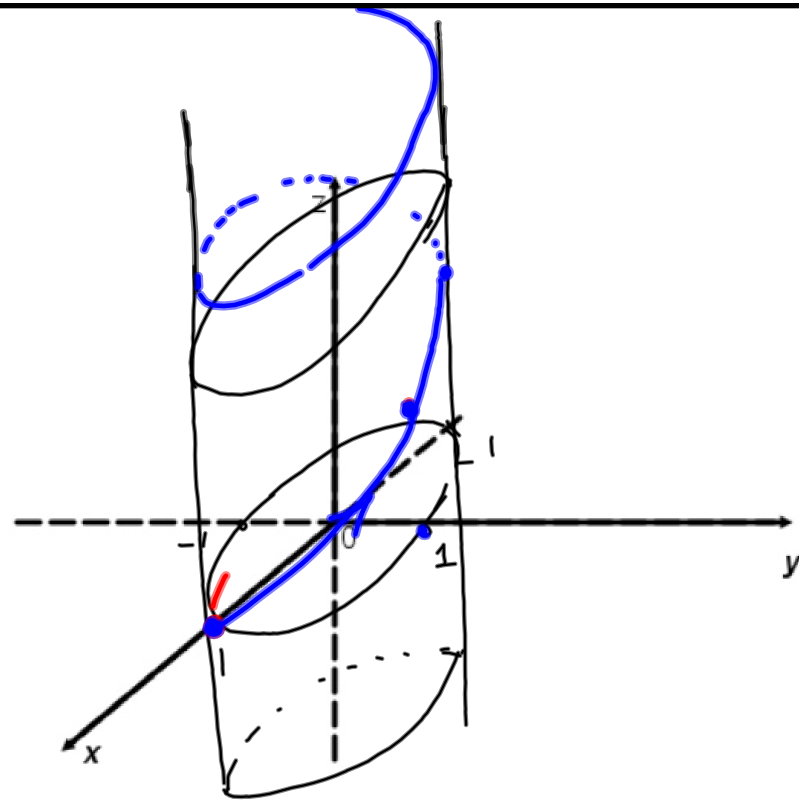
$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \Rightarrow \underbrace{x^2 + y^2 = 1}_{\text{circular cylinder}}$$

$$z = t$$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, \frac{\pi}{2} \rangle$

$\vec{r}(\pi) = \langle -1, 0, \pi \rangle$

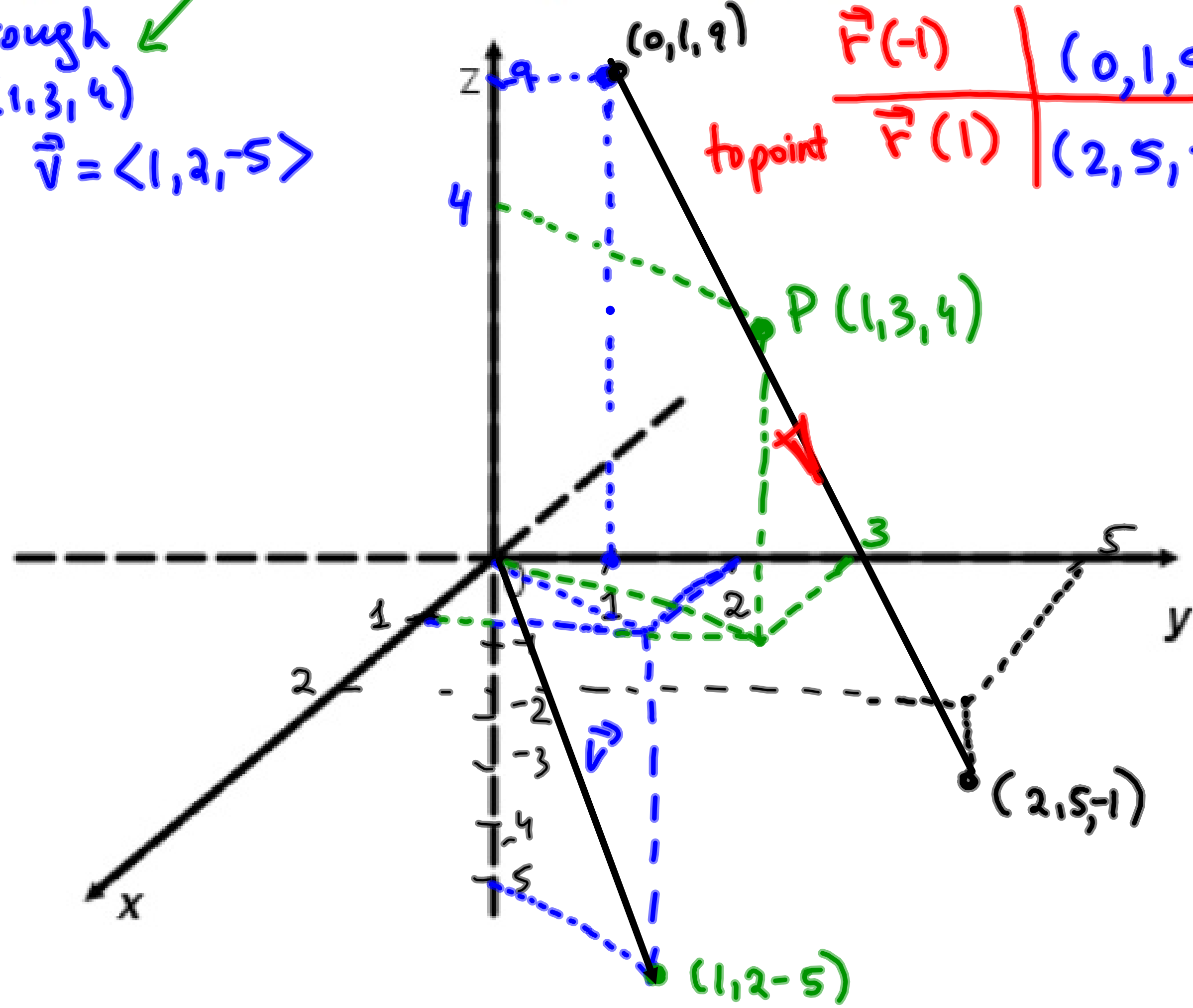


(e) $\mathbf{r}(t) = \langle 1+t, 3+2t, 4-5t\rangle$, $-1 \le t \le 1$.

line segment from point

$\vec{r}(-1)$	$(0,1,9)$
to point $\vec{r}(1)$	$(2,5,-1)$

line through the point $P(1,3,4)$ parallel to $\vec{v} = \langle 1,2,-5\rangle$



EXAMPLE 2. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

$x = \sin t$ (1)

$y = 2\cos t$ (2)

$z = \sqrt{3}\sin t$ (3)

$$x^2+y^2+z^2 \overset{(1),(2)\&(3)}{=} \sin^2 t + 4\cos^2 t + 3\sin^2 t$$

$$= 4\sin^2 t + 4\cos^2 t = 4(\sin^2 t + \cos^2 t)$$

$$= 4 \cdot 1 = 4$$

So, the curve belongs to the sphere

$$x^2+y^2+z^2 = 4.$$

$\Downarrow$

(1) and (3) imply that $z = \sqrt{3}\,x$ and this equation of plane through origin. Thus the given curve belongs to this plane as well.

So, the given curve is a line of intersection between sphere centered at origin and with radius 2 and plane through the origin.

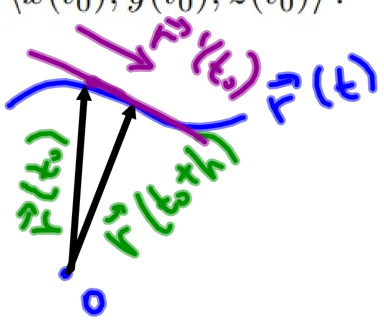
Thus the given curve is a circle centered at $(0,0,0)$ with radius 2.

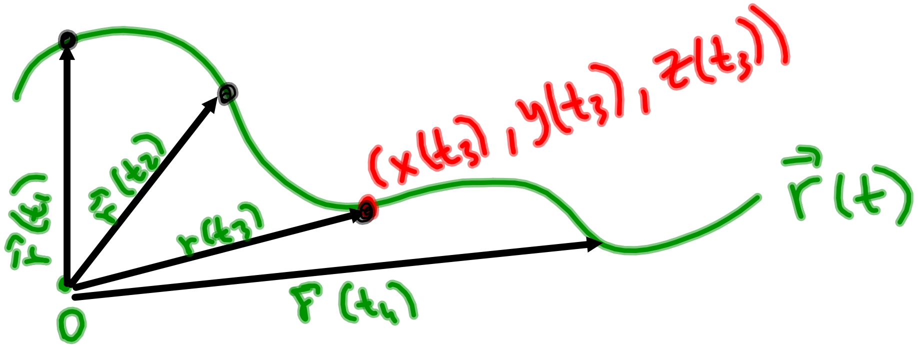
13.2 Derivatives of Vector Functions

The derivative $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}$$

if the limit exists. The derivative $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$.





THEOREM 3. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

EXAMPLE 4. *Given* $\mathbf{r}(t) = (1+t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$

$\vec{r}'(t) = \langle ((1+t)^2)', (e^t)', (\sin 3t)' \rangle$

$= \langle 2(1+t), e^t, 3\cos 3t \rangle$

given by $\vec{r}(t)$

(b) *Find a tangent vector to the curve at* $t = 0$.

$\vec{v} = \vec{r}'(0) = \langle 2(1+0), e^0, 3\cos 3\cdot 0 \rangle = \langle 2, 1, 3 \rangle$

(c) *Find a tangent line to the curve at* $t = 0$.

First find position at time $t=0$:

$\vec{r}(0) = \langle (0+1)^2, e^0, \sin(3\cdot 0) \rangle = \langle 1, 1, 0 \rangle$

So, the tangent passes through the point $(1,1,0)$ and parallel to the vector $\langle 2,1,3 \rangle$ (see part (b))

$x = 1+2t,\ y = 1+t,\ z = 3t$

(c′) *Find a tangent line to the curve at the point* $(1, 1, 0)$.

Find t_0 such that $\vec{r}(t_0) = \langle 1,1,0 \rangle$.

Then calculate $\vec{r}'(t_0)$ (= tangent vector).