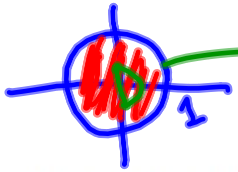
14.1: Functions of Several Variables

Consider the following formulas:

$$z = 2 - x - 4y \qquad D = \mathbb{R}^2 \tag{1}$$

$$z = x^2 + y^2 \qquad D = \mathbb{R}^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \qquad D = \mathbb{R}^2 \tag{3}$$



$$z = \sqrt{1 - x^2 - y^2} \qquad D = \{(x,y) \mid 1 - x^2 - y^2 \geq 0\} \tag{4}$$

$x^2 + y^2 \leq 1$

ordered pair of independent variables (x, y) (input)	$\longrightarrow$	$f(x,y)$	$\longrightarrow$	dependent variable z (output)

subset

DEFINITION 1. *Let $D \subseteq \mathbb{R}^2$. A* **function** *f* **of two variables** *is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.*

The set D is the **domain** *of f and the* **range** *of f is the set of values that f takes on, that is* $\{f(x, y) | (x, y) \in D\}$.

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

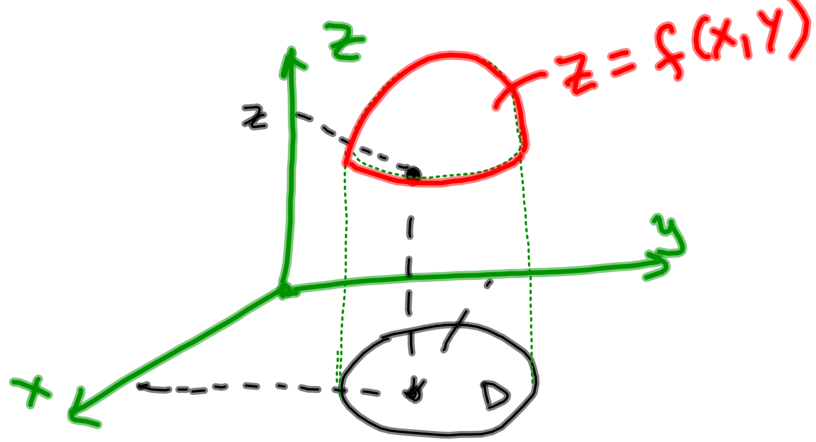
• **GRAPH of** $f(x, y)$.

Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

DEFINITION 3. *The* **graph** *of* f *with domain* D *is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), \ (x, y) \in D\right\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.



EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $z = 2 - x - 4y$ plane

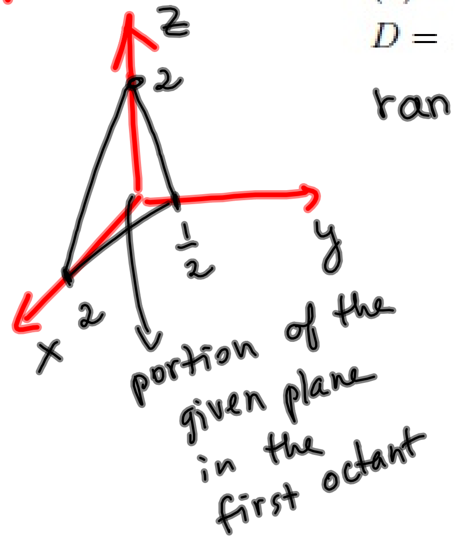
$D = \mathbb{R}^2$

range $= \mathbb{R}$

$(0,0,2)$

$(2,0,0)$

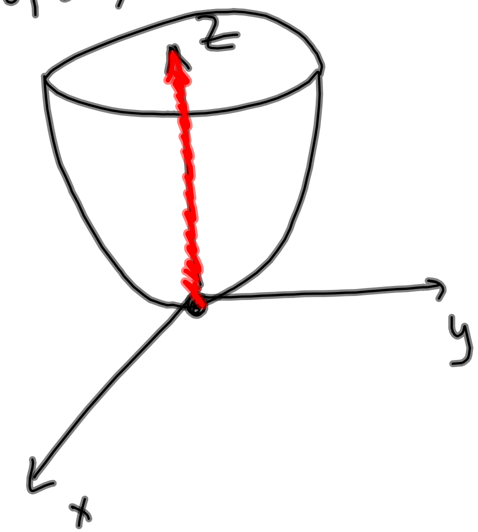
$(0,\frac{1}{2},0)$



(2) $z = x^2 + y^2$ circular paraboloid

$D = \mathbb{R}^2$

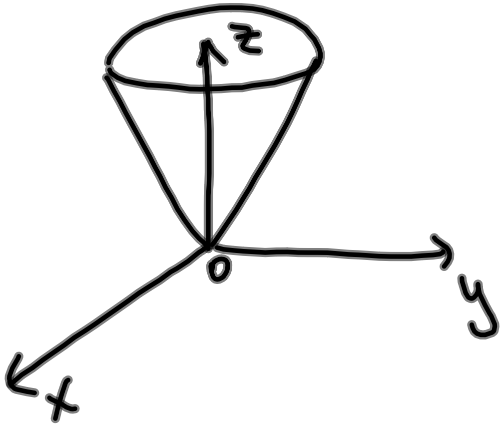
range $= [0, \infty)$



(3) $z = \sqrt{x^2 + y^2}$ upper part of circular cone

$D =$ $\mathbb{R}^2$

range = $[0, \infty)$

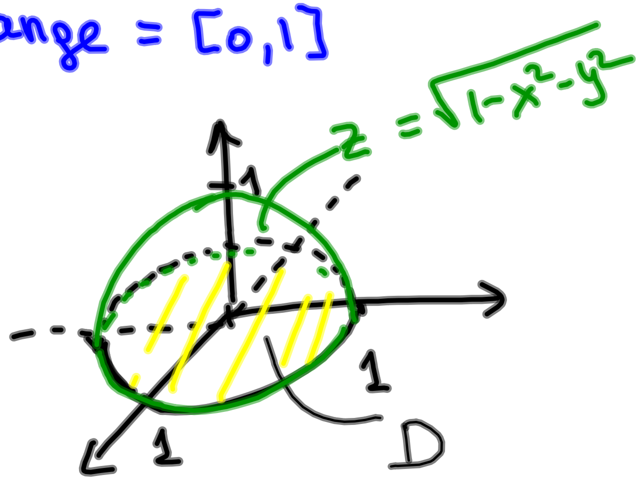


(4) $z = \sqrt{1 - x^2 - y^2}$ upper semi sphere with radius 1 centered at origin.

$D = \{(x,y) \mid 1 - x^2 - y^2 \geq 0\}$

$= \{(x,y) \mid x^2 + y^2 \leq 1\}$

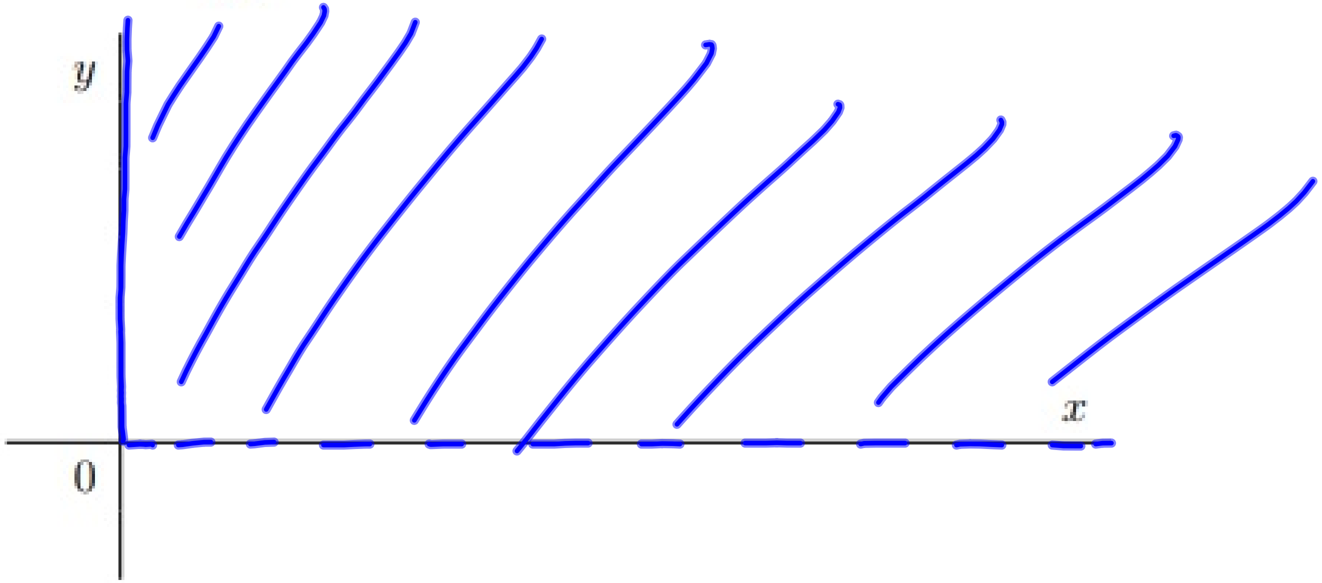
range = $[0,1]$



EXAMPLE 5. *Sketch the domain of each of the following:* functions

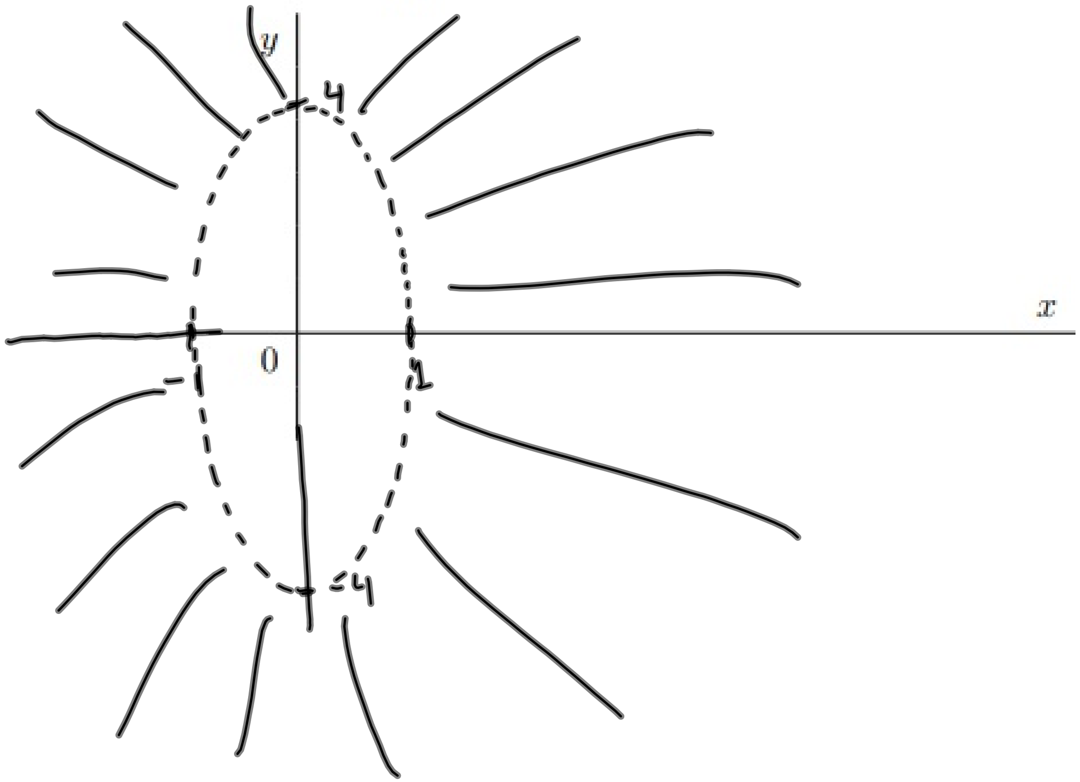
(a) $\overset{f(x,y)}{\cancel{z}} = \sqrt{x} - \frac{5}{\sqrt{y}}$

$D = \{(x,y) : x \geq 0 \text{ and } y > 0\}$



(b) $f(x,y) = \ln(x^2 + \frac{y^2}{16} - 1)$

$D = \{(x,y) : \; x^2 + \frac{y^2}{16} - 1 > 0\}$



• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

$z = f(x, y)$

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

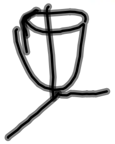
DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

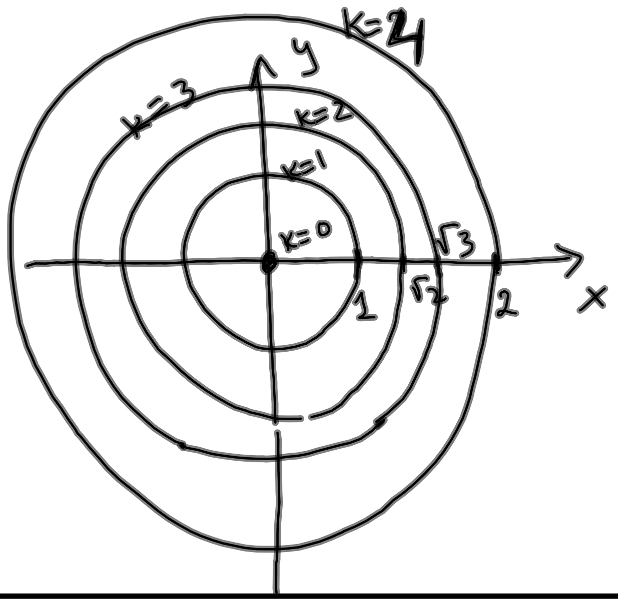
EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values* $k = 0, 1, 2, 3, 4$:



(2) $z = x^2 + y^2$ circular paraboloid

$$x^2+y^2=k \quad (k \geq 0)$$

		radius
$k=0$	$(0,0)$	
$k=1$	$x^2+y^2=1$	1
$k=2$	$x^2+y^2=2$	$\sqrt{2}$
$k=3$	$x^2+y^2=3$	$\sqrt{3}$
$k=4$	$x^2+y^2=4$	2

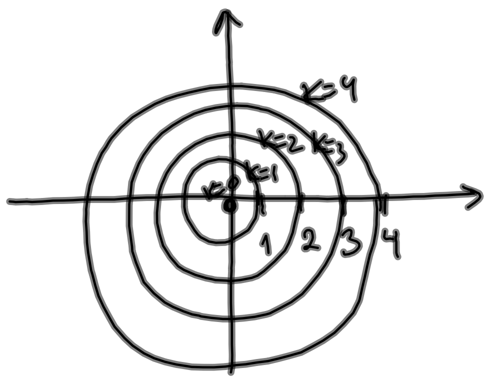


(3) $z = \sqrt{x^2+y^2}$ upper circular cone

$$\sqrt{x^2+y^2} = k \quad (k \geq 0)$$

$$x^2+y^2=k^2$$

radius	
$k=0$	$(0,0)$
$k=1$	$x^2+y^2=1$
$k=2$	$x^2+y^2=4$
$k=3$	$x^2+y^2=9$
$k=4$	$x^2+y^2=16$



- **Functions of three variables.**

DEFINITION 8. *Let* $D \subset \mathbb{R}^3$. *A* **function** f **of three variables** *is a rule that assigns to each ordered pair* (x, y, z) *in* D *a unique real number denoted by* $f(x, y, z)$.

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

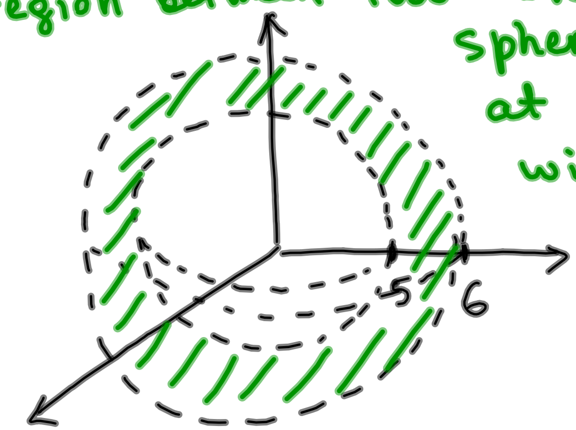
Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. *Find the domain of the following function:*

$$f(x,y,z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2+y^2+z^2-25}}.$$

$D(f) = \{(x,y,z) : 36 - x^2 - y^2 - z^2 > 0 \text{ and } x^2+y^2+z^2-25>0\}$

region between two concentric spheres centered at origin with radius 5 and 6.



$36 - x^2 - y^2 - z^2 = 0$

$x^2+y^2+z^2 = 36$

$x^2+y^2+z^2 = 25$

Test points $(0,0,0)$

$(0, 5.5, 0)$ ✓

$(7, 0, 0)$

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function* $\cancel{y}$ $f(x,y,z)$ $= x^2 + y^2 - z$.

Level surfaces: $f(x, y, z) = k$

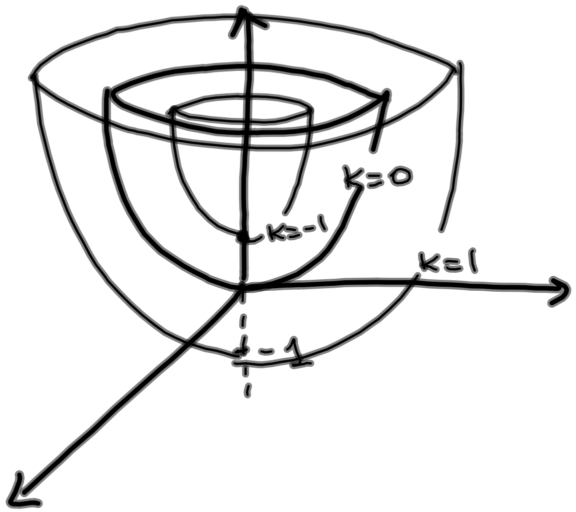
$$x^2 + y^2 - z = k$$

$$z + k = x^2 + y^2$$ circular paraboloids

$k=0$, $z = x^2+y^2$

$k=1$, $z+1 = x^2+y^2$

$k=-1$, $z-1 = x^2+y^2$



REMARK 11. For any function there exist a unique level curve (surface) through given point!!!