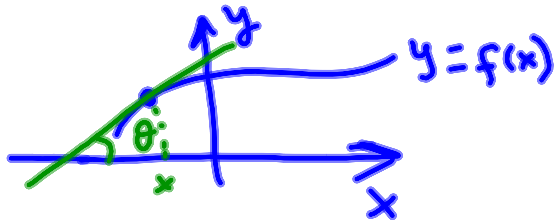
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\tan\theta = f'(x)$



14.3: Partial Derivatives

DEFINITION 1. *If f is a function of two variables, its* **partial derivatives** *are the functions f_x and f_y defined by*

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

∂ partial

$\partial \neq d$

Conclusion: $f_x(x, y)$ represents the *rate of change* of the function $f(x, y)$ as we change x and hold y fixed while $f_y(x, y)$ represents the rate of change of $f(x, y)$ as we change y and hold x fixed.

Notations for partial derivatives: If $z = f(x, y)$, we write

$z = f(x, y)$

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Notations for partial derivatives: If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y =$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x,y)$:

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

EXAMPLE 3. *If* $f(x,y) = x^3 + y^5 e^x$ *find* $f_x(0,1)$ *and* $f_y(0,1)$.

$f_x(x,y) = 3x^2 + y^5 e^x \Rightarrow f_x(0,1) = 3\cdot 0^2 + 1^5 \cdot e^0 = \boxed{1}$

$f_y(x,y) = 0 + 5y^4 e^x \Rightarrow f_y(0,1) = 5 \cdot 1^4 \cdot e^0 = \boxed{5}$

EXAMPLE 4. *Find all of the first order partial derivatives for the following functions:*

(a) $z(x,y) = x^3 \sin(xy)$

$$z_x = \frac{\partial z}{\partial x} \overset{\text{Product Rule}}{=} (x^3)' \cdot \sin(xy) + x^3 \frac{\partial}{\partial x}(\sin(xy))$$

Chain Rule

$$= 3x^2 \sin(xy) + x^3 \cos(xy) \underbrace{\frac{\partial (xy)}{\partial x}}_{= y}$$

$$= 3x^2 \sin(xy) + x^3 y \cos(xy)$$

$$z_y = \frac{\partial}{\partial y}(x^3 \sin(xy)) = x^3 \cos(xy) \underbrace{\frac{\partial}{\partial y}(xy)}_{= x} = x^4 \cos(xy)$$

(c) $u(x,y,z) = ye^{xyz}$

$$u_x = y \frac{\partial}{\partial x}(e^{xyz}) = y e^{xyz} \underbrace{\frac{\partial}{\partial x}(xyz)}_{yz} = \boxed{y^2 z e^{xyz}}$$

$$u_y \overset{\text{Product Rule}}{=} 1 \cdot e^{xyz} + y \frac{\partial}{\partial y}(e^{xyz}) = e^{xyz} + y e^{xyz} \underbrace{\frac{\partial}{\partial y}(xyz)}_{xz}$$

$$= \boxed{e^{xyz} + xyze^{xyz}}$$

$u_z = y^2 x e^{xyz}$ (By symmetry between x and z).

EXAMPLE 5. *The temperature at a point (x, y) on a flat metal plate is given by*

$$T(x,y) = \frac{80}{1+x^2+y^2},$$

where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(1, 2)$ in the y-direction. This is $T_y(1,2)$

$$T_y = \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{80}{1+x^2+y^2}\right) = 80 \cdot \left(-\frac{1}{(1+x^2+y^2)^2}\right) \cdot \frac{\partial}{\partial y}(1+x^2+y^2)$$

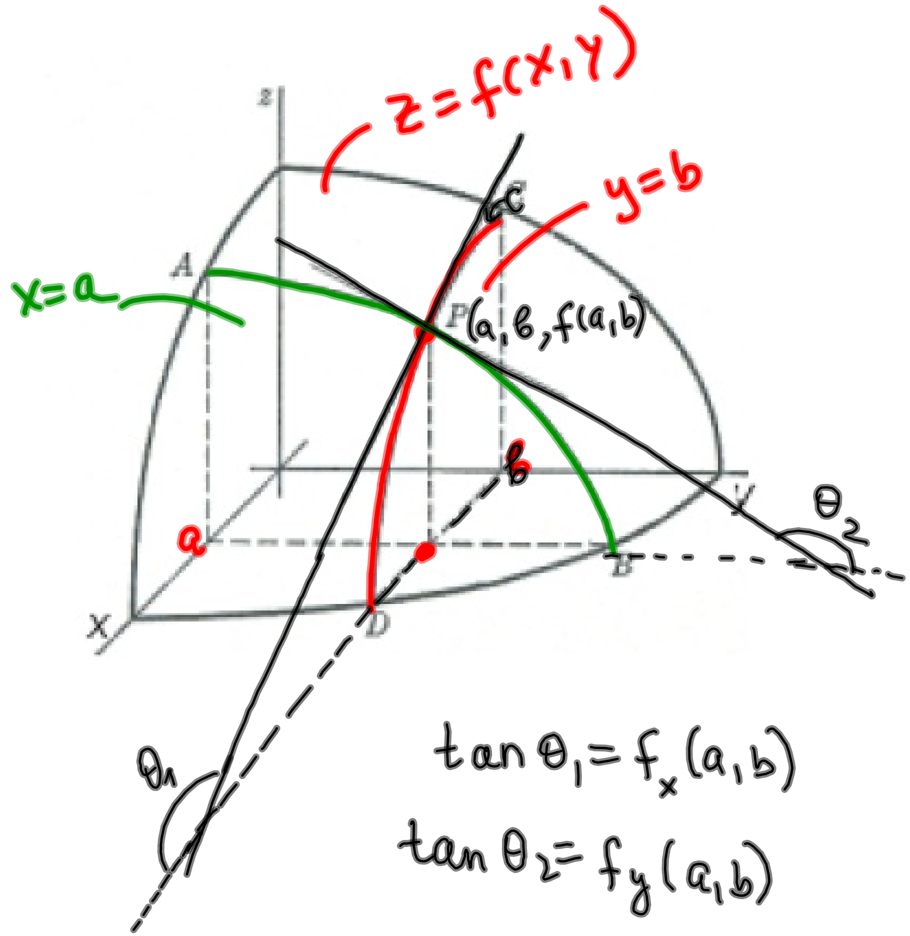
$$= \frac{-80 \cdot 2y}{(1+x^2+y^2)^2}$$

$$T_y(1,2) = \frac{-80 \cdot 2 \cdot 2}{(1+1^2+2^2)^2} = -\frac{80 \cdot \cancel{4}^{1}}{\cancel{36}_{9}} = -\frac{80}{9}\ {}^{\circ}\mathrm{C}/\mathrm{m}$$

Geometric interpretation of partial derivatives: Partial derivatives are the *slopes of traces*:

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point (a,b).

- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at (a,b).



EXAMPLE 6. *If* $f(x,y)=\sqrt{4-x^2-4y^2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*

$$f_x = \frac{1}{2\sqrt{4-x^2-4y^2}} \cdot \frac{\partial}{\partial x}(4-x^2-4y^2)$$

$$= \frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2-4y^2}} \Rightarrow f_x(1,0) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$f_y = \frac{-8y}{2\sqrt{4-x^2-4y^2}} = -\frac{4y}{\sqrt{4-x^2-4y^2}} \Rightarrow f_y(1,0) = 0$$

$$\tan\theta_1 = f_x(1,0) = -\frac{\sqrt{3}}{3}$$

$$\theta_1 = \frac{5\pi}{6}$$

$$\tan\theta_2 = f_y(1,0) = 0$$

$$\theta_2 = 0$$

(tangent is horizontal)

$f(x,y)=\sqrt{4-x^2-4y^2}$

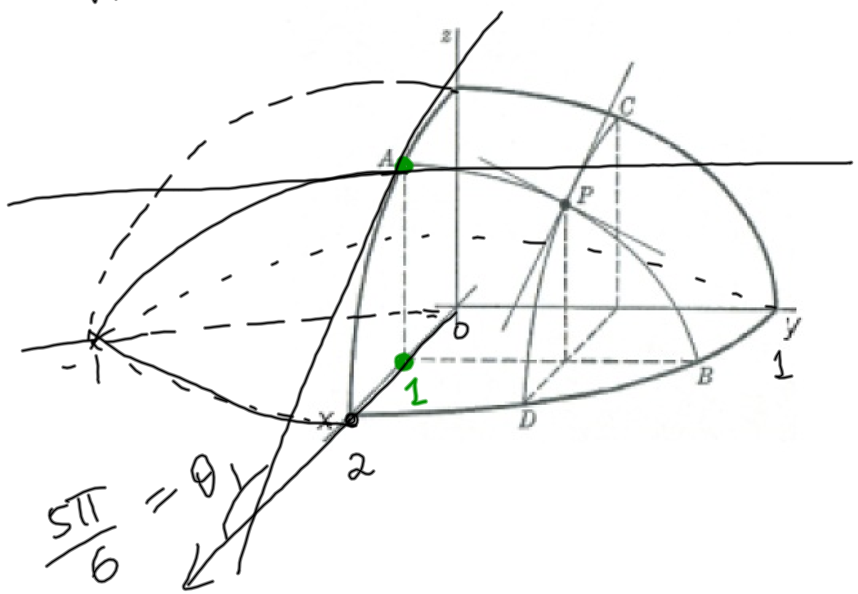
$z=\sqrt{4-x^2-4y^2}$

$z^2=4-x^2-4y^2\ ,\ z\geq 0$

$x^2+4y^2+z^2=4\ ,\ z\geq 0$

$\frac{x^2}{4}+y^2+\frac{z^2}{4}=1,\ z\geq 0$

upper semi ellipsoid



Derivative of higher orders (2, 3, 4, ...)

Higher derivatives: Since both of the first order partial derivatives for $f(x, y)$ are also functions of x and y, so we can in turn differentiate each with respect to x or y. We use the following notation:

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial z^2}
\end{aligned}
$$

mixed derivatives

EXAMPLE 7. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

first order partial derivatives $\left\{ \begin{array}{l} f_x = 20y^2e^{4x} + 2x\sin(x^2) \\ f_y = 3y^2 + 10ye^{4x} \end{array} \right.$

$$f_{xx} = \frac{\partial}{\partial x}\left(20y^2e^{4x} + \underline{2x}\,\underline{\sin(x^2)}\right)$$

$$= 80y^2e^{4x} + 2\sin(x^2) + 4x^2\cos(x^2)$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(20y^2e^{4x} + 2x\sin(x^2)\right)$$

$$= 40ye^{4x}$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(3y^2 + 10ye^{4x}\right) = 40ye^{4x}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(3y^2 + 10ye^{4x}\right)$$

$$= 6y + 10e^{4x}.$$

Clairaut's Theorem. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions f_{yyx}, f_{xyy} and f_{yxy} are continuous then

$$f_{yyx} = f_{xyy} = f_{yxy}$$

EXAMPLE 8. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

(a) f_{xy} $= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\sin(xy+z)\frac{\partial}{\partial y}(xy+z)\right)$

$= \frac{\partial}{\partial x}\left(\underbrace{-x\sin(xy+z)}_{f_y}\right)$

$= -\sin(xy+z) - xy\cos(xy+z)$

(b) f_{zxy} $= f_{xyz} = \frac{\partial}{\partial z}(f_{xy}) = \frac{\partial}{\partial z}\left(-\sin(xy+z) - xy\cos(xy+z)\right)$

$= -\cos(xy+z) + xy\sin(xy+z)$