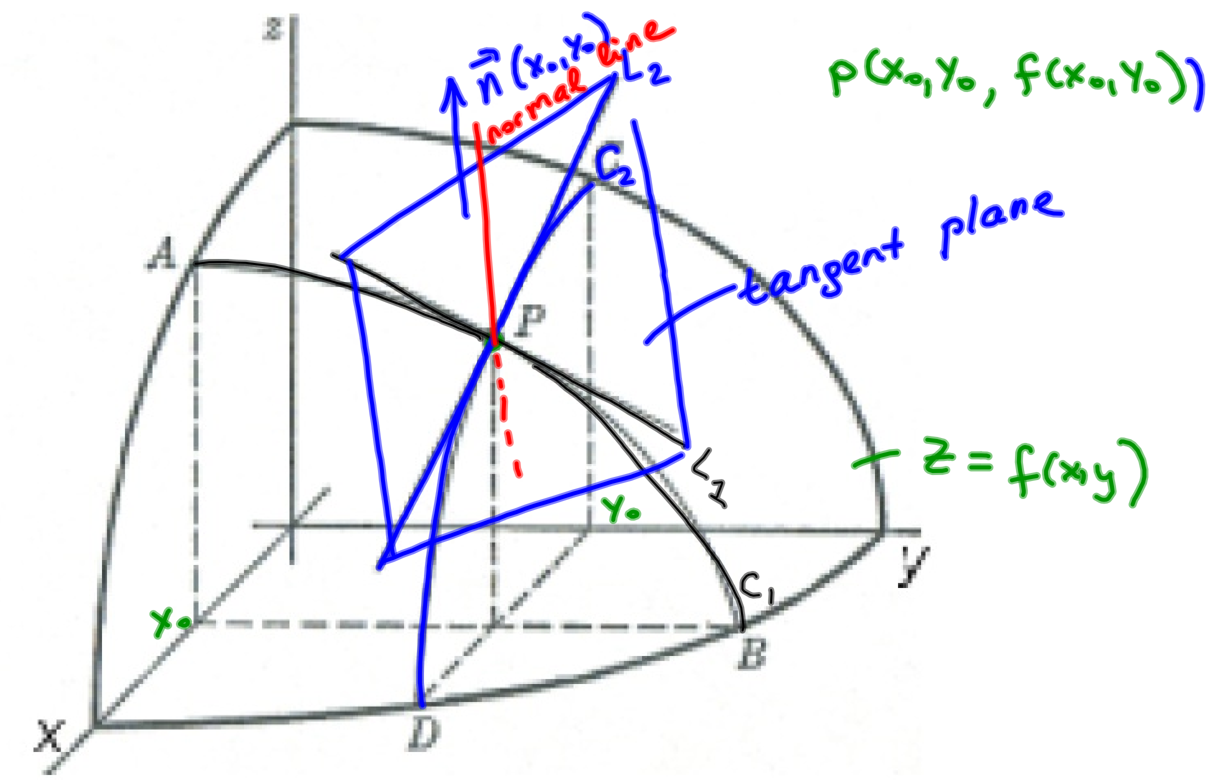
to the graph $z=f(x,y)$

14.4: Tangent Planes and Linear Approximation

Suppose that $f(x, y)$ has continuous first partial derivatives and a surface S has equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x, y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x, y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x, y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



THEOREM 1. *An equation of the tangent plane to the graph of the function* $z = f(x, y)$ *at the point* $P(x_0, y_0, f(x_0, y_0))$ *is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

*CONCLUSION:*A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle.$$

The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to the above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. It follows that this normal line can be expressed parametrically as

component of $\vec{n}$

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

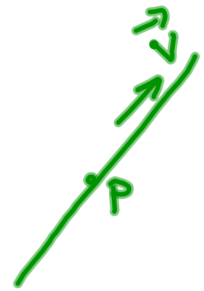
component of P

$$x = x_0 + f_x(x_0, y_0)t$$
$$y = y_0 + f_y(x_0, y_0)t$$
$$z = f(x_0, y_0) - t$$

Ex 3 Find param. equation for the normal line to the surface

$$z = e^{4y}\sin(4x)$$

at the point $P(\frac{\pi}{8}, 0, 1)$.



Solution The direction vector of the normal line is parallel to the normal to the tangent plane at P.

So, $\vec{v} = \vec{n}(\frac{\pi}{8}, 0) = \langle z_x(\frac{\pi}{8}, 0), z_y(\frac{\pi}{8}, 0), -1 \rangle$

$z_x = 4e^{4y}\cos(4x)$

$z_y = 4e^{4y}\sin(4x)$

$\vec{v} = \langle 4e^{0}\cos(4\cdot\frac{\pi}{8}), 4e^{0}\sin(4\cdot\frac{\pi}{8}), -1 \rangle$

$\vec{v} = \langle 4\cos\frac{\pi}{2}, 4\sin\frac{\pi}{2}, -1 \rangle = \langle 0, 4, -1 \rangle$

So, the normal line is

$x = \frac{\pi}{8} + 0\cdot t$

$y = 0 + 4\cdot t$

$z = 1 + (-1)\cdot t$

or $\boxed{x = \frac{\pi}{8},\ y = 4t,\ z = 1 - t \quad (t \in \mathbb{R})}$

EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function* $z = x^2 + y^2 + 8$ *at the point* $(1, 1)$.

x_0, y_0

Tangent point

$z(1,1) = 1^2 + 1^2 + 8 = 10$

$P(1, 1, 10)$

normal vector to the plane

$\vec{n}(x,y) = \langle z_x, z_y, -1 \rangle = \langle 2x, 2y, -1 \rangle$

normal at P:

$\vec{n}(1,1) = \langle 2, 2, -1 \rangle$

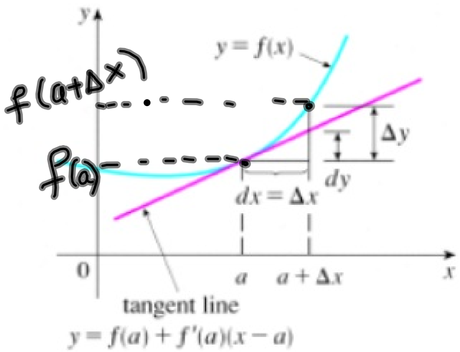
Tangent plane at (1,1) :

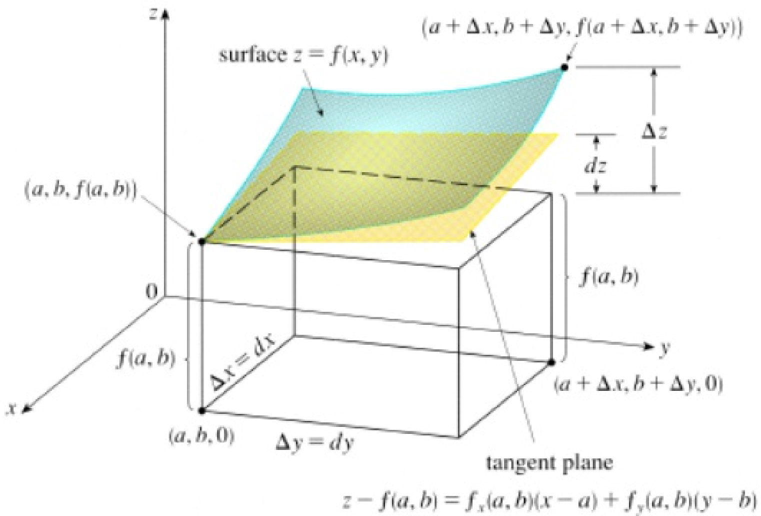
$2(x-1) + 2(y-1) + (-1)(z-10) = 0$

$2x + 2y - z + 6 = 0$

Differentials. Given $z = f(x, y)$. If Δx and Δy are given increments of $x = a$ and $y = b$ respectively, then the corresponding **increment** of z is

$$\Delta z(a, b) = f(a + \Delta x, b + \Delta y) - f(a, b). \tag{1}$$





The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **total differential**) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$

$z = f(x,y)$

$df = f_x dx + f_y dy$

FACT: $\Delta z \approx \mathrm{d}z$.

This implies:

$$\boxed{f(a+\Delta x, b+\Delta y) \approx f(a,b) + \mathrm{d}z(a,b)}$$

or

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + \underbrace{f_x^{(a,b)}\Delta x + f_y^{(a,b)}\Delta y}_{\text{differential}}$$

EXAMPLE 4. *Use differentials to find an approximate value for* $\sqrt{1.03^2 + 1.98^3}$.

$f(x,y) = \sqrt{x^2 + y^3}$

$a = 1, \quad b = 2$

$\Delta x = 0.03, \quad \Delta y = -0.02$

$$\sqrt{1.03^2 + 1.98^3} = f(1.03, 1.98) =$$

$$= f(\underbrace{1}_{a} + \underbrace{0.03}_{\Delta x}, \underbrace{2}_{b} \underbrace{- 0.02}_{\Delta y}) \approx$$

$$\approx f(1,2) + f_x(1,2)(0.03) + f_y(1,2)(-0.02)$$

Note that

$$f(1,2) = \sqrt{1^2 + 2^3} = 3$$

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^3}\right) = \frac{x}{\sqrt{x^2+y^3}}$$

$$f_x(1,2) = \frac{1}{\sqrt{1^2+2^3}} = \boxed{\frac{1}{3}}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^3}\right) = \frac{3y^2}{2\sqrt{x^2+y^3}}$$

$$f_y(1,2) = \frac{3\cdot 2^2}{2\cdot 3} = \boxed{2}$$

So,

$$\sqrt{1.03^2+1.98^3} \approx 3 + \frac{1}{3}\cdot 0.03 + 2\cdot(-0.02)$$

$$= 3 + 0.01 - 0.04 = 3 - 0.03 = \boxed{2.97}$$

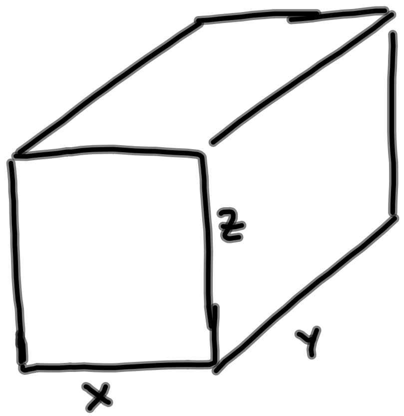
By Calculator: 2.97040

If $u = f(x, y, z)$ then the differential $\mathrm{d}u$ at the point $(x, y, z) = (a, b, c)$ is defined in terms of the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ of the independent variables:

$$\mathrm{d}u(a, b, c) = f_x(a, b, c)\mathrm{d}x + f_y(a, b, c)\mathrm{d}y + f_z(a, b, c)\mathrm{d}z.$$

$$\Delta u(a,b,c) \approx du(a,b,c)$$

EXAMPLE 5. *The dimensions of a closed rectangular box are measured as* 80 *cm,* 60 *cm and* 50 *cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*



Surface area

$$S = 2(xz + yz + xy)$$

Find $\Delta S(80, 60, 50)$

and $\Delta x = \Delta y = \Delta z = 0.2$

$dx = dy = dz = 0.2$

We know that $\Delta S(80,60,50) \approx dS(80,60,50)$

So, first find dS:

$$dS = S_x\,dx + S_y\,dy + S_z\,dz$$

$$dS = (S_x + S_y + S_z)dx =$$

$$= (2(z+y) + 2(z+x) + 2(x+y))dx =$$

$$= 2(2z + 2x + 2y)dx = 4(x+y+z)dx$$

$$dS(80,60,50) = 4(80+60+50)0.2$$

$$= 4 \cdot 190 \cdot 0.2 = 8 \cdot 19 = 152\ cm^2$$

$$\Delta S(80,60,50) \approx 152\ cm^2$$

A function $f(x, y)$ is **differentiable** *at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).*

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a **linear function** of two variables.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$

The function $L(x, y)$ is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the tangent plane approximation of f at (a, b).

If f_x and f_y are continuous, then f is differentiable